Geometric realisation of ~~regular~~ constructible polygraphs

Journées LHC:

Logique, homotopie, & catégories

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Introduction

- A central goal in this work is to connect the following domains:

Higher-dimensional (algebraic) rewriting	Directed (algebraic) topology

- These are both approaches to directed calculations/processes.
- In each, we have notions of homotopy:

Squier & coherence	Natural homotopy

We would like to relate these two invariants.

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- More generally, these approaches complement each other:

Generators Finiteness conditions	Directed CW complexes Cellular homotopy
Algebraic description of directed homotopy Topological Squier's theorem	Natural homotopy Topology and continuity

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- Hadzihasanovic's constructible directed complexes provide a combinatorial presentation of higher categories.
- Additionally, they are realisable as regular CW complexes.

1. Constructible directed complexes and polygraphs

2. CW complexes and directed topological cells

3. Realisation (work in progress. . .)

Constructible directed complexes and polygraphs

Oriented thin posets

N.B. All results and definitions in this section are from recent works by Amar Hadzihasanovic.

What are the properties required of a "face poset" for higher categorical cells?

- Recall that the Hasse diagram $\mathcal{H}P$ of P is the directed graph with

$$\mathcal{H}P_0 = P \qquad \mathcal{H}P_1 = \{(y, x) \mid y \text{ covers } x\}.$$

Oriented thin posets

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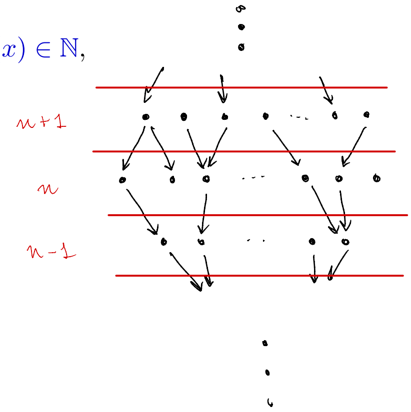
- A notion of dimension : graded.
    - to each $x \in P$ we associate $\dim(x) \in \mathbb{N}$,



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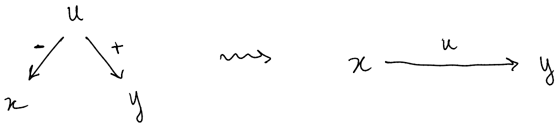

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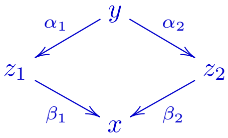
- A manifold-like condition : thinness.
  - We ask that elements intersect nicely, both geometrically and w.r.t. orientation



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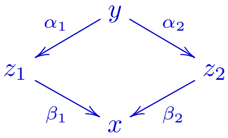

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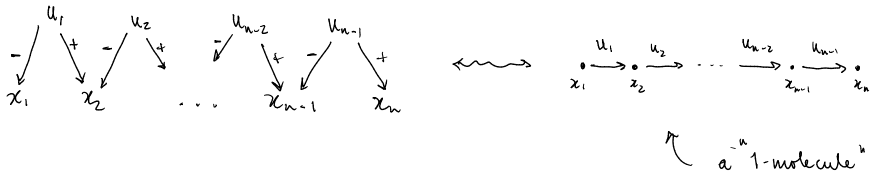
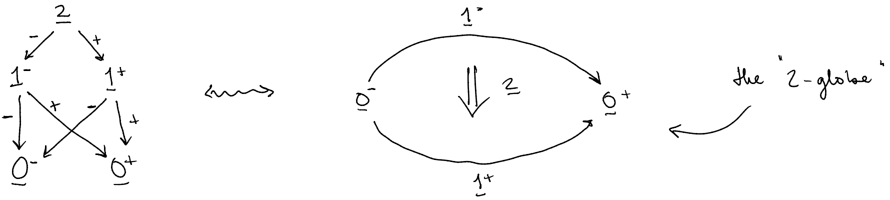
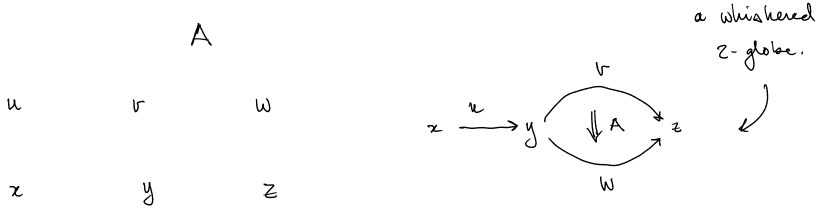
∂^- ∂^+

- P is then an oriented thin poset, and we define input/output borders, dimension/purity of closed subsets…

Examples







Constructible directed complexes

Let P be an oriented thin poset. For each $n \in \mathbb{N}$, we single out a a poset $(C\mathcal{M}ol_n P, \sqsubset)$ of n-dimensional closed pure subsets of P. These are called constructible n-molecules.

→ 0-molecules are 0-dimensional elements.

→ n-molecules are either:

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- Let $\mathbf{ogPos}_{in}$ denote the category of oriented graded posets and closed inclusions.

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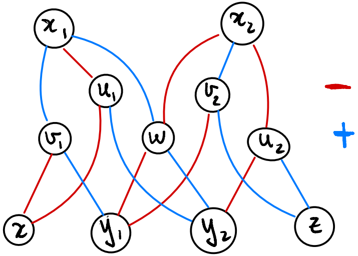
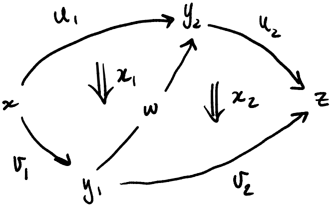
Proposition

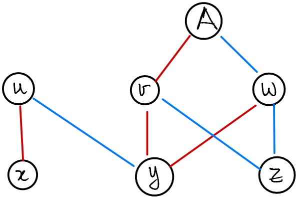
A constructible directed complex is the colimit of the diagram of inclusions of its atoms.

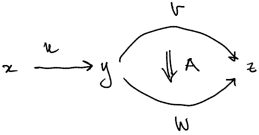
Examples





↑ a non-atomic constructible 2-molecule.





This is not a constructible molecule...

Constructible polygraphs

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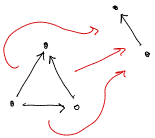
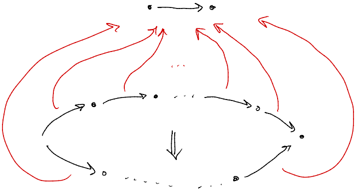
- A constructible polygraph is a presheaf

$$X : \mathbf{CAtom}^{op} \longrightarrow \mathsf{Set}.$$

An element $x \in X(U)$, for $\dim(U) = n$ is a n-cell of shape U.

- This is similar to simplicial complexes, but with more complex (globular) base shapes.





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- Constructible polygraphs and morphisms of presheaves form a category $\mathbf{CPol}$.

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- Via Steiner's theory of directed complexes, Hadzihasanovic shows that there exists a functor

$$\mathbf{CDCpx} \longrightarrow \omega\mathbf{Cat}$$

- The restriction of this functor to **CAtom** is denoted by $(-)_\omega$.
- The left Kan extension of the Yoneda embedding **CAtom** $\rightarrow$ **CPol** yields a functor

$$\begin{aligned} \mathbf{CPol} &\longrightarrow \omega\mathbf{Cat} \\ X &\longmapsto X_\omega := \int^{U \in \mathbf{CAtom}} U_\omega \times X(U) \end{aligned}$$

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- Recently, a theorem was in fact found to be a conjecture:

Conjecture

For a constructible polygraph X, the omega category X_ω admits the structure of a polygraph whose n-dimensional generators are indexed by n-cells of X.

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- This yields spaces with the structure of CW complex:

Theorem

For a constructible polygraph X, the space $|-|$ admits the structure of a CW complex whose generating n-cells are indexed by n-cells of X.

CW complexes and directed topological cells

CW complexes

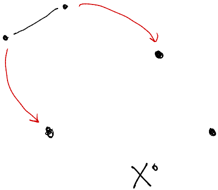
- Morally, a CW complex X is a space built from topological cells homeomorphic to n-balls D^n.
  - Start with a discrete set of points to form the 0-skeleton X^0.
  - Having constructed the $(n-1)$-skeleton X^{n-1}, place the borders of n-balls via attaching maps

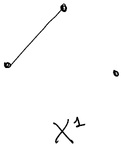
$$\phi_x : S^{n-1} \longrightarrow X^{n-1}.$$

  - Form the n-skeleton

$$X^n = X^{n-1} \coprod_x e_x^n / \{\phi_x\}_x$$

    where the quotient indicates the identification of the images of the attaching maps ϕ_x.





Globular CW complexes

- A notion of directed CW complex was introduced by P. Gaucher and E. Goubault. The directed topological cells they used are built using the following construction.
- Consider the globe functor:

$$Glob : \mathsf{Top} \longrightarrow \mathsf{PoTop}$$
$$X \longmapsto (X \times I/\sim, \leq)$$

where $(x,t) \sim (x',t')$ iff $t = t' \in \{0,1\}$ and

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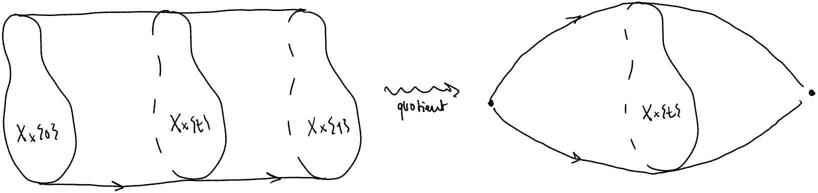

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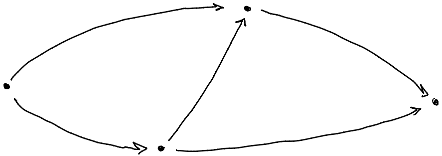
- These are not flexible enough to accommodate glueing along partial borders:



we don't obtain a space which is dihomeomorphic to $\vec{D}^2$...

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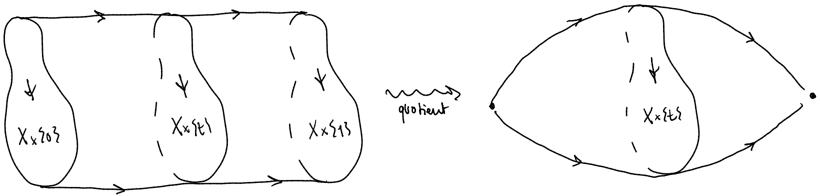

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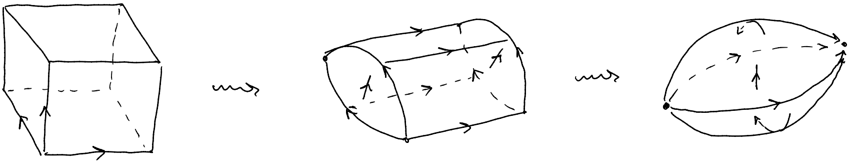
- These are thus quotients of the n-cube:

$$\begin{aligned}I^n = \vec{\vec{D}}^1 \times I^{n-1} \to \vec{\vec{D}}^2 \times I^{n-2} \to \cdots\\ \cdots \to \vec{\vec{D}}^k \times I^{n-k} \to \cdots \to \vec{\vec{D}}^{n-1} \times I \to \vec{\vec{D}}^n.\end{aligned}$$



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$$\cdots \to \vec{\vec{D}}^k \times I^{n-k} \to \cdots \to \vec{\vec{D}}^{n-1} \times I \to \vec{\vec{D}}^n.$$

- In this view, points in $\vec{\vec{D}}^n$ are represented by n-tuples $(t_n, \dots, t_1)$, the quotient being given by

$$(t_n, \dots, t_1) \sim (t'_n, \dots, t'_1) \quad \Longleftrightarrow \quad \exists i, t_i = t'_i = b \text{ for } b \in \{0, 1\}, \text{ and } \forall j < i, t_j = t'_j.$$

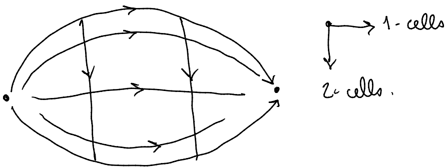
Directed globes

- We obtain a nice definition for borders; for any $k \leq n-1$ set

$$\partial_k^-(\overrightarrow{\overrightarrow{D}}^n) := \{\overline{(t_n, \ldots, t_{k+2}, 0, t_k, \ldots, t_1)} | t_i \in I\}$$
$$\partial_k^+(\overrightarrow{\overrightarrow{D}}^n) := \{\overline{(t_n, \ldots, t_{k+2}, 1, t_k, \ldots, t_1)} | t_i \in I\}.$$

  We have $\partial_k^\alpha(\overrightarrow{\overrightarrow{D}}^n) \cong \overrightarrow{\overrightarrow{D}}^k$.

- These cells are 1-dimensionally oriented in each dimension; the variable t_k corresponds to "cells of dimension k".



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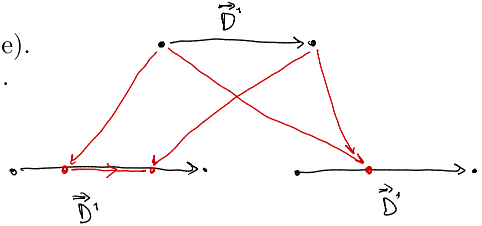

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- Suppose that maps $[\overrightarrow{D}^{n-1}, \overrightarrow{D}^{n-1}]$ have been constructed. A map $i : \overrightarrow{D}^n \to \overrightarrow{D}^n$ is given by:
  - a monotonic map $\iota : I \to I$,
  - a continuous family of maps

$$j_{(-)} : I \longrightarrow [\overrightarrow{D}^{n-1}, \overrightarrow{D}^{n-1}]$$

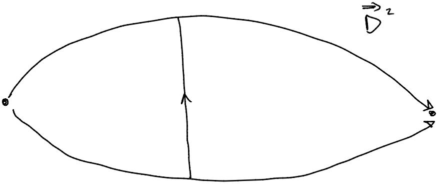
  such that j_0 and j_1 are constant maps and j_t is non degenerate for $t \in]0,1[$.

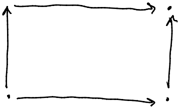
  Then i is the quotient map of

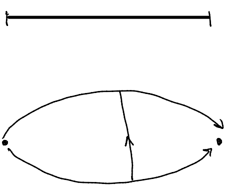
$$\begin{aligned} \overrightarrow{D}^{n-1} \times I &\longrightarrow \overrightarrow{D}^n \\ (x,t) &\longmapsto (j_t(x), \iota(t)). \end{aligned}$$

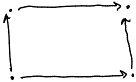
  These maps are called subglobes.

Structural maps





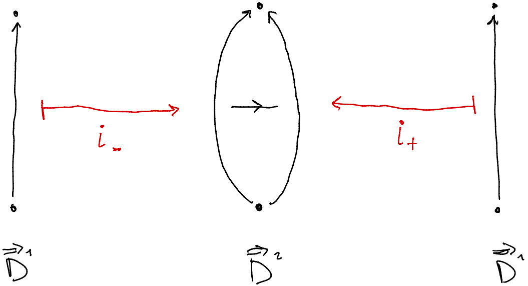


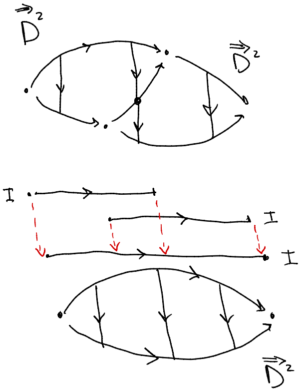


Structural maps

- We generate maps $\overrightarrow{\overrightarrow{D}}^k \to \overrightarrow{\overrightarrow{D}}^n$ with subglobes and total borders:

$$i_\alpha : \overrightarrow{\overrightarrow{D}}^{n-1} \cong \partial^\alpha \overrightarrow{\overrightarrow{D}}^n \longrightarrow \overrightarrow{\overrightarrow{D}}^n$$
$$\overline{(t_{n-1}, \dots t_1)} \longmapsto \overline{(b, t_{n-1}, \dots, t_1)},$$





Realisation (work in progress. . .)

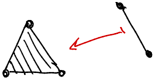
Directed geometric realisation

When defining a geometric realisation of presheaves on a category S, we can realise each shape $s \in S_0$

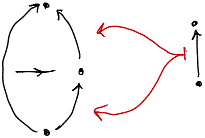
$\Delta \longrightarrow \text{Top}$

$\underline{n} \longmapsto$ standard n-simplex

⇝ We need a topological interpretation of face maps



In the case of constructible complexes, the face maps are more complicated; they are "partial":



We need to already have an interpretation of $(n-1)$-dimensional CDCpx's in order to understand how $(n-1)$-dimensional atoms are included in n-dimensional ones.

Directed geometric realisation

$$\left(D_P : \underline{P} \longrightarrow \mathrm{CAtom},\quad x \longmapsto \mathrm{cl}(x). \right)$$

We initialise the construction:

$U \overset{R}{\longmapsto} \vec{D}^0 = \{*\}$; $P \longmapsto |P| := \operatorname{colim}_{\underline{P}} (R \circ D_P)$

a 0-mol. — a 0-dimensional CDCpx

If we know how to realise $(n-1)$-atoms (and thereby $(n-1)$-dim'l CDCpx's), we extend:

$U \overset{R}{\longmapsto} \vec{D}^n$

n-atom

$V \xrightarrow{\varphi} U$

$(n-1)$ atom

$\exists! \alpha \in \{-,+\}$ s.t. $V \xrightarrow{\varphi'} \partial^\alpha U \to U$, with $\partial^\alpha U \cong \vec{D}^{n-1}$, $V \cong \vec{D}^{n-1}$, and $\varphi = (\partial^\alpha U \to U) \circ \varphi'$

$R(\varphi) = i_\alpha \circ |\varphi'|$

Now, if P is an n-dimensional CDCpx, again $|P| := \operatorname{colim}_{\underline{P}} (R \circ D_P)$.

We need to show that $|U| \cong \vec{D}^n$ whenever U is a constructible n-molecule.

Thank You

Coming soon:

Algebraic Rewriting Seminar

- Organisers:
  - Benjamin Dupont (bdupont@math.univ-lyon1.fr)
  - Cyrille Chenavier (cyrille.chenavier@unilim.fr)
  - Cameron Calk (cameron.calk@lix.polytechnique.fr)
- Modalities:
  - Thursdays 16h - 18h (bi)weekly.
  - Seminar held virtually.
  - Expected to start in March
- Feel free to contact us with any questions or comments.

See you soon!

Graded posets

N.B. All results and definitions in this section are from recent works by Amar Hadzihasanovic. Let P be a poset.

- Let $x, y \in P$. We say that y covers x if

$$x < y \quad \text{and} \quad \forall y' \in P, x < y' \leq y \Rightarrow y' = y.$$

- The Hasse diagram $\mathcal{H}P$ of P is the directed graph with

$$\mathcal{H}P_0 = P \qquad \mathcal{H}P_1 = \{(y, x) \mid y \text{ covers } x\}.$$

- Denote by $P_\perp$ the poset P with a least element $\perp$ added.
- P is graded if for every x, all paths from x to $\perp$ have the same length l_x. We set

$$\dim(x) := l_x - 1$$

  and $P^{(n)} := \{x \in P | dim(x) = n\}$.

Orientation, closures and borders

Let P be a graded poset.

- An orientation of P is a map

$$o : \mathcal{H}P_1 \longrightarrow \{-,+\}.$$

  In this case we say that P is an oriented graded poset.
- Recall that for $U \subseteq P$, the closure of U is defined as

$$cl(U) = \{x \in P | \exists z \in U, x \leq z\},$$

  and U is closed if $U = cl(U)$.
- We set $\dim U = \max\{\dim(x) | x \in U\}$ and $\dim(\emptyset) = -1$.
- A closed n-dimensional subset U is pure if $U = cl(U^{(n)})$.

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- We set $\dim U = \max\{\dim(x) | x \in U\}$ and $\dim(\emptyset) = -1$.
- Closed subsets inherit gradedness and orientation from P. For a closed n-dimensional subset U, we set
$$\Delta^\alpha U := \{ x \in U^{(n-1)} \mid \forall y \in U, \text{ if } y \text{ covers } x, \text{ then } o(y,x) = \alpha\}$$
$$\partial^\alpha U := cl(\Delta^\alpha U) \cup \{ x \in U \mid \forall y \in U, x \leq y \Rightarrow dim(y) < n\},$$
$$\Delta U := \Delta^+ U \cup \Delta^- U, \qquad \partial U := \partial^+ U \cup \partial^- U.$$
- $\partial_n^- U$ (resp. $\partial_n^+ U$) is the input (resp. output) boundary of U.

Orientation, closures and borders

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- We set $\dim U = \max\{\dim(x) | x \in U\}$ and $\dim(\emptyset) = -1$.
- An inclusion of oriented thin posets is a closed embedding of posets that is compatible with the orientation maps. We denote the category of oriented graded posets and inclusions by $\mathbf{ogPos}_{in}$

Oriented thin posets

Let P be an oriented graded poset.

- A consequence of gradedness is that for $x \leq y$, all paths from x to y in $\mathcal{H}P$ have the same length $\dim(y) - \dim(x)$. This is the length of the interval $[x, y]$.

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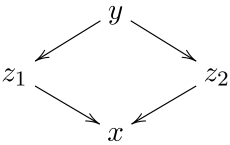
- P is thin if any interval $[x, y]$ of length 2 in $P_{\perp}$ contains exactly 4 elements:



Oriented thin posets

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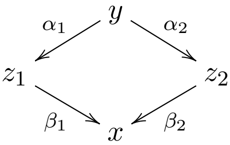
- A consequence of gradedness is that for $x \leq y$, all paths from x to y in $\mathcal{H}P$ have the same length $\dim(y) - \dim(x)$. This is the length of the interval $[x, y]$.
- If P is oriented, we extend o to $P_\perp$ by setting $o(x, \perp) = +$ for all minimal $x \in P$.
- P is an oriented thin poset if for any interval $[x, y]$ of length 2 we have thinness:



and additionally $\alpha_1\beta_1 = -\alpha_2\beta_2$.

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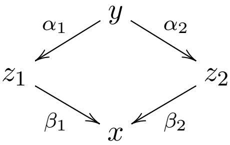

  and additionally $\alpha_1\beta_1 = -\alpha_2\beta_2$.
- Examples:

Constructible directed complexes

Let P be an oriented thin poset. For each $n \in \mathbb{N}$, we single out a a poset $(C\mathcal{M}ol_n P, \sqsubset)$ of n-dimensional closed subsets of P. These are called constructible n-molecules.

- $C\mathcal{M}ol_0 P := \{\{x\ \} |\dim(x) = 0\}$ with the discrete order.

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- $C\mathcal{M}ol_0 P := \{\{x\} | \dim(x) = 0\}$ with the discrete order.
- For a pure n-dimensional subset U, we have $U \in C\mathcal{M}ol_n P$ if $\partial^+ U, \partial^- U \in C\mathcal{M}ol_{n-1} P$ and either
  - U has a greatest element, or
  - $U = U_1 \cup U_2$ such that
    - U_1, U_2 partition the maximal elements of U,
    - $U_1 \cap U_2 = \partial^+ U_1 \cap \partial^- U_2 \in C\mathcal{M}ol_{n-1} P$, and
    - $\partial^- U_1 \sqsubseteq \partial^- U$, $\partial^+ U_2 \sqsubseteq \partial^+ U$, $U_1 \cap U_2 \sqsubseteq \partial^+ U$, and $U_1 \cap U_2 \sqsubseteq \partial^- U$.
  - A constructible molecule with a largest element is called an atom.

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- Constructible n-molecules are globular in the sense that

$$\partial^\alpha(\partial^+ U) = \partial^\alpha(\partial^- U) \quad \text{and} \quad \partial^+ U \cap \partial^- U = \partial(\partial^\alpha U).$$

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- An oriented thin poset P is called a constructible directed complex if $cl(x)$ is a constructible atom for all $x \in P$.
- The full subcategory of $\mathbf{ogPos}_{in}$ consisting of constructible directed complexes is denoted by **CDCpx**.