Languages of Higher-Dimensional Automata via Pomset Objects

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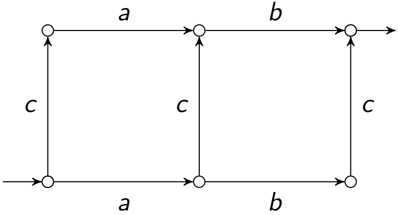
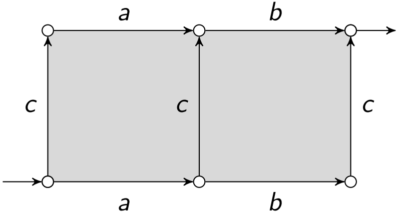
Higher-Dimensional Automata



an automaton

$$L(A) = \{abc, acb, cab\}$$

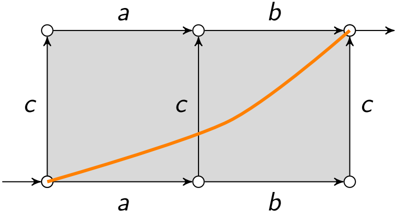
Higher-Dimensional Automata



a higher-dimensional automaton (HDA)

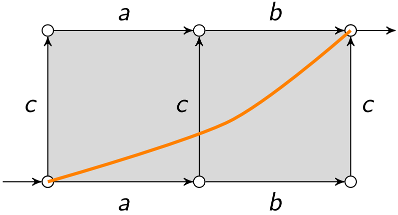
$$L(A) = \left\{ \left(\begin{array}{c} a \longrightarrow b \\ c \end{array} \right), \dots \right\}$$

Higher-Dimensional Automata

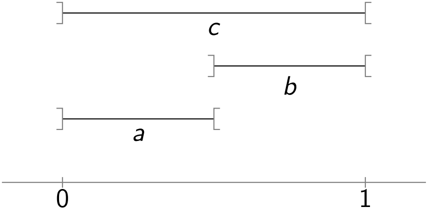


executions are directed paths (d-paths)

Higher-Dimensional Automata

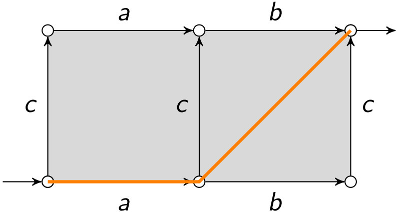


interval arrangement of d-path:

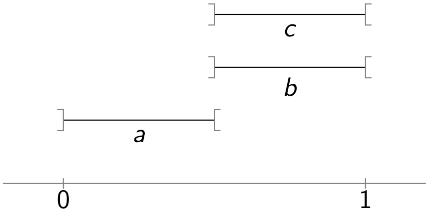


$$\left(\begin{array}{c} a \longrightarrow b \\ c \end{array}\right)$$

Higher-Dimensional Automata

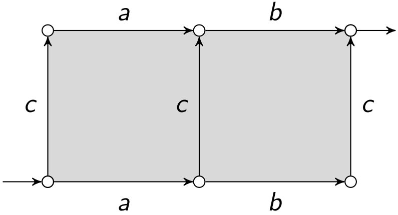


different d-path:



$$\left(\begin{tikzcd} a \arrow[r] \arrow[dr] & b \\ & c \end{tikzcd}\right)$$

Higher-Dimensional Automata

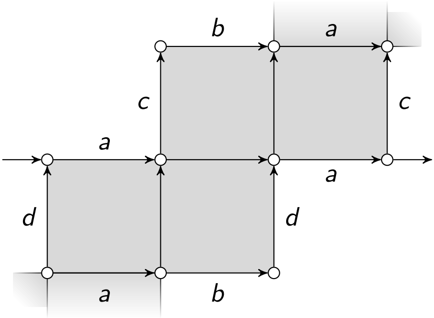


$$L(A) = \left\{ \left(\begin{tikzcd} a \arrow[rr] & & b \\ & c & \end{tikzcd}\right), \left(\begin{tikzcd} a \arrow[r] \arrow[dr] & b \\ & c \end{tikzcd}\right), \left(\begin{tikzcd} a \arrow[r] & b \\ c \arrow[ur] & \end{tikzcd}\right), \right.$$

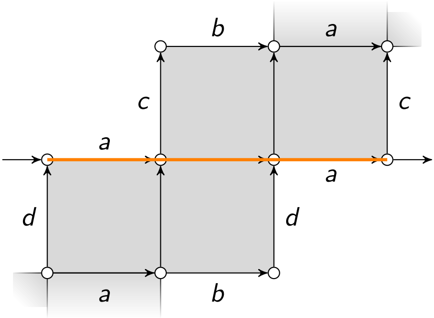
$$\left. \left(\begin{tikzcd} a \arrow[r] & b \arrow[r] & c \end{tikzcd}\right), \left(\begin{tikzcd} a \arrow[r] & c \arrow[r] & b \end{tikzcd}\right), \left(\begin{tikzcd} c \arrow[r] & a \arrow[r] & b \end{tikzcd}\right) \right\}$$

$$= \left\{ \left(\begin{tikzcd} a \arrow[rr] & & b \\ & c & \end{tikzcd}\right) \right\}\Big\downarrow$$

Another Example

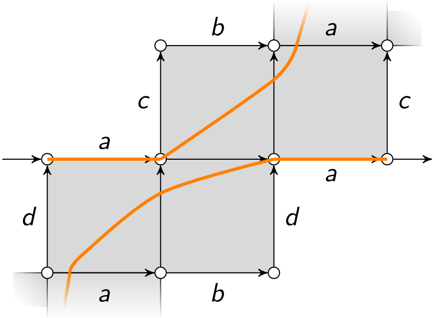


Another Example



$$L(A)=\left\{ \left(a \longrightarrow b \longrightarrow a \right), \left(\begin{tikzcd} a \arrow[r] \arrow[dr] & b \arrow[r] \arrow[dr] & a \arrow[r] & b \arrow[r] & a \\ & c \arrow[r] \arrow[urr] & d \arrow[urr] & & \end{tikzcd}\right), \dots \right\}\downarrow$$

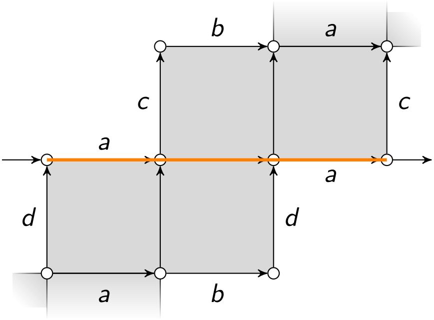
Another Example



$$L(A) = \left\{ \left(a \longrightarrow b \longrightarrow a \right), \left(\begin{tikzcd} a \arrow[r] \arrow[dr] & b \arrow[r] \arrow[dr] & a \arrow[r] & b \arrow[r] & a \\ & c \arrow[r] \arrow[urr] & d \arrow[urr] & & \end{tikzcd} \right), \dots \right\}\downarrow$$

Tracks

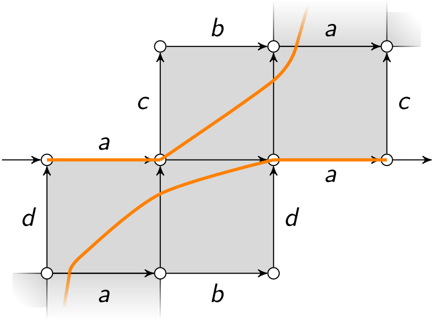
Sequences of cells connected at boundaries / faces:



a, b, a

Tracks

Sequences of cells connected at boundaries / faces:



$$a, \begin{pmatrix} b \\ c \end{pmatrix}, \begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} a \\ d \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}, a$$

- correspondence d-paths – tracks

1. Motivation
2. Precubical Sets
3. Tracks
4. Conclusion

The Large Precube Category

The "Ziemiański" category $\boxdot$:

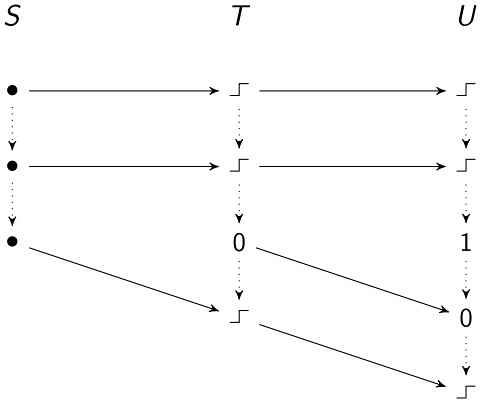
- objects totally ordered sets S
- $\boxdot(S,T)=\{(f,\varepsilon)\mid f:S\to T \text{ order injection}, \varepsilon:T\to\{0,\lrcorner\!\ulcorner,1\}, f(S)=\varepsilon^{-1}(\lrcorner\!\ulcorner)\}$
- $(g,\zeta)\circ(f,\varepsilon):S\to T\to U=(g\circ f,\eta)$:

$$\eta(u)=\begin{cases}\varepsilon(g^{-1}(u)) & \text{for } u\in g(T),\\ \zeta(u) & \text{otherwise.}\end{cases}$$

- 0: not started; $\lrcorner\!\ulcorner$: executing; 1: terminated (*cf.* Chu spaces)

$\Longrightarrow$ in an injection $f:S\to T$, events in $f(S)$ are executing; the others are "at the boundary"

Composition in ⊡



$(g,\zeta)\circ(f,\varepsilon) : S \to T \to U = (g\circ f,\eta)$:

$$\eta(u) = \begin{cases} \varepsilon(g^{-1}(u)) & \text{for } u \in g(T), \\ \zeta(u) & \text{otherwise.} \end{cases}$$

Context

augmented presimplex category Δ	large aug. presimplex category ◬
objects $\{1 < \dots < n\}$ for $n \geq 0$	objects totally ordered sets
morphisms order injections	morphisms order injections
skeletal	isos are unique

$\Delta \hookrightarrow \text{◬}$ equivalence with unique left inverse

(augmented) precube category $\square$	large (aug.) precube category $\boxdot$
objects $\{0,1\}^n$ for $n \geq 0$	objects totally ordered sets
morphisms index-order injections	morphisms (f, ε)
skeletal	isos are unique

$\square \hookrightarrow \boxdot$ equivalence with unique left inverse

- presimplicial sets: $\mathsf{Set}^{\Delta^{\mathrm{op}}}$ or $\mathsf{Set}^{\text{◬}^{\mathrm{op}}}$; makes no difference
- precubical sets: $\mathsf{Set}^{\square^{\mathrm{op}}}$ or $\mathsf{Set}^{\boxdot^{\mathrm{op}}}$; makes no difference

Precubical Sets

In elementary terms: a precubical set (pc-set) is a graded set $X = \bigsqcup_{n \geq 0} X_n$ (disjoint union) together with elementary face maps

$$\delta^{\nu}_{i,n} : X_n \to X_{n-1} \qquad (i \in \{1, \dots, n\}, \nu \in \{0, 1\})$$

which are usually written without the extra index "$_n$" and satisfy the precubical identities

$$\delta^{\nu}_{i} \delta^{\mu}_{j} = \delta^{\mu}_{j-1} \delta^{\nu}_{i} \qquad (i < j)$$

Labelings and Events

- the labeling object on a finite set Σ: the presheaf $!\Sigma(S) = \mathsf{Set}(S, \Sigma)$
- $!\Sigma_n = \Sigma^n$; $\delta_i^\nu((a_1, \dots, a_n)) = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$
- a labeled precubical set: $\lambda : X \to !\Sigma$

- the event object on a finite set E: the presheaf $!!E(S) = \mathsf{Inj}(S, E)$ (*injective* functions)
- $!!E_n = \{(e_1, \dots, e_n) \in E^n \mid \forall i \neq j : e_i \neq e_j\}$; δ_i^ν as above
- an event consistent precubical set: $\exists f : X \to !!E$

Not all precubical sets are event consistent, but we're only interested in those which are.

Lemma

X is event consistent iff the equivalence $\sim_{\mathrm{ev}}$ generated on X_1 by $\{(\delta_1^0 x, \delta_1^1 x), (\delta_2^0 x, \delta_2^1 x) \mid x \in X_2\}$ satisfies $\forall x \in X_2 : \delta_1^0 x \not\sim_{\mathrm{ev}} \delta_2^0 x$.

- $E_X = X_1 / \sim_{\mathrm{ev}}$: the universal events of X

Higher-Dimensional Automata

Definition

An higher-dimensional automaton, over a finite alphabet Σ, is a labeled event-consistent precubical set $\lambda : X \to !\Sigma$ together with subsets $I, F \subseteq X$ of initial and final cells.

- Labeling factors uniquely through universal events: $\lambda = \lambda^{\text{ev}} \circ \text{ev} : X \to !!E_X \to !\Sigma$
- For the purpose of this talk, we will mostly ignore the labeling.

1. Motivation
2. Precubical Sets
3. Tracks
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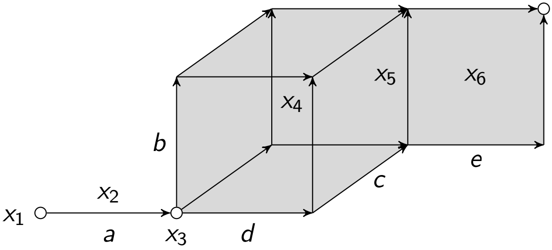
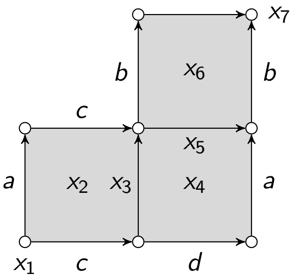


A track in X: a sequence $\rho = (x_1, \dots, x_m)$, $m \geq 1$, s.t. for all i:

$$x_i = \delta_A^{0,\dots,0} x_{i+1} \qquad \text{or} \qquad x_{i+1} = \delta_A^{1,\dots,1} x_i$$

for some A

Poset Labels of Tracks

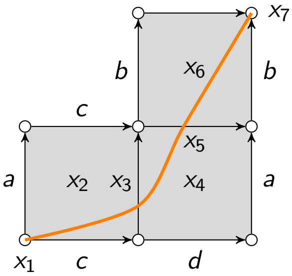


Informally, the label of a track: gluing of maximal cells along minimal cells

$$\begin{aligned}\rho &= x_1,\ x_2,\ x_3,\ x_4,\ \ x_5,\ \ x_6,\ \ x_7\\ &= \emptyset, \begin{pmatrix} a \\ c \end{pmatrix}, a, \begin{pmatrix} a \\ d \end{pmatrix}, d, \begin{pmatrix} b \\ d \end{pmatrix}, \emptyset\end{aligned}$$

$$\ell(\rho) = \left(\begin{tikzcd} a \arrow[r] & b \\ c \arrow[ur] \arrow[r] & d \end{tikzcd}\right)$$

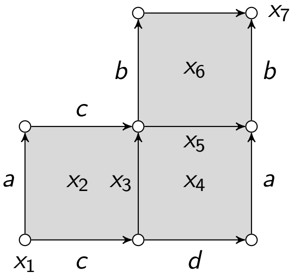
Properties



- For every track there is a d-path with the same label.
- For every d-path there is a track with the same label.
- Labels of tracks are interval orders.

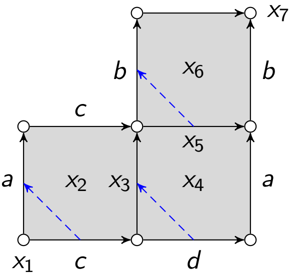
- Track $(x_1, \dots, x_m)$ accepting if $x_1 \in I$ (initial) & $x_m \in F$ (final)
- Language of HDA X: $L(X) = \{\ell(\rho) \mid \rho \text{ accepting}\}$

But Wait, There's More



Precubical sets are locally ordered (due to event consistency):

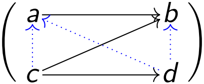
But Wait, There's More



Precubical sets are locally ordered (due to event consistency):

$$\rho = \emptyset, \begin{pmatrix} a \\ \uparrow \\ c \end{pmatrix}, a, \begin{pmatrix} a \\ \uparrow \\ d \end{pmatrix}, d, \begin{pmatrix} b \\ \uparrow \\ d \end{pmatrix}, \emptyset$$

$\ell(\rho) =$



From Posets to Tracks

An iposet $(P, <, \dashrightarrow)$: two partial orders $<$, $\dashrightarrow$ s.t. $< \cup \dashrightarrow$ is *total*.

- All $<$-antichains in P are totally $\dashrightarrow$-ordered.

Define relation $\prec$ on $\{0, {\lrcorner\!\ulcorner}, 1\}$ by $\prec = \{(0,0), ({\lrcorner\!\ulcorner}, 0), (1,0), (1, {\lrcorner\!\ulcorner}), (1,1)\}$

Definition

The poset object of an iposet P is the precubical set $\square^P$, as follows:

- $\square_k^P = \{x : (P, <) \to (\{0, {\lrcorner\!\ulcorner}, 1\}, \prec) \mid |x^{-1}({\lrcorner\!\ulcorner})| = k\}$
- for $x \in \square_k^P$ and $x^{-1}({\lrcorner\!\ulcorner}) = \{p_1 \dashrightarrow \cdots \dashrightarrow p_k\}$,

$$\delta_i^\nu(x)(p) = \begin{cases} \nu & \text{for } p = p_i \\ x(p) & \text{for } p \neq p_i \end{cases}$$

Example

Recall $\prec = \{(0,0), (\lrcorner\ulcorner, 0), (1,0), (1, \lrcorner\ulcorner), (1,1)\}$. Let $P = \left(\begin{tikzcd} a \arrow[r] & b \\ c \arrow[u, dotted, blue] \arrow[r] \arrow[ur] & d \arrow[u, dotted, blue] \arrow[ul, dotted, blue] \end{tikzcd}\right)$.

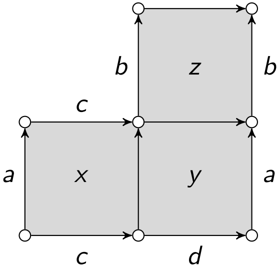
- $\square_2^P$: $x = \left(\begin{tikzcd} \lrcorner\ulcorner \arrow[r] & 0 \\ \lrcorner\ulcorner \arrow[r] \arrow[ur] & 0 \end{tikzcd}\right)$ $y = \left(\begin{tikzcd} \lrcorner\ulcorner \arrow[r] & 0 \\ 1 \arrow[r] \arrow[ur] & \lrcorner\ulcorner \end{tikzcd}\right)$ $z = \left(\begin{tikzcd} 1 \arrow[r] & \lrcorner\ulcorner \\ 1 \arrow[r] \arrow[ur] & \lrcorner\ulcorner \end{tikzcd}\right)$
- $\delta_1^\nu x = \left(\begin{tikzcd} \lrcorner\ulcorner \arrow[r] & 0 \\ \nu \arrow[r] \arrow[ur] & 0 \end{tikzcd}\right)$ $\delta_2^\nu x = \left(\begin{tikzcd} \nu \arrow[r] & 0 \\ \lrcorner\ulcorner \arrow[r] \arrow[ur] & 0 \end{tikzcd}\right)$ etc.

Example

Recall $\prec \, = \{(0,0), (\lrcorner\!\ulcorner, 0), (1,0), (1,\lrcorner\!\ulcorner), (1,1)\}$. Let

$$P = \left(\begin{tikzcd} a \arrow[r] & b \\ c \arrow[u, dotted, blue] \arrow[ur] \arrow[r] & d \arrow[u, dotted, blue] \arrow[ul, dotted, blue] \end{tikzcd} \right).$$

- $\color{red}\square_2^P$: $x = \left(\begin{tikzcd} \lrcorner\!\ulcorner \arrow[r] & 0 \\ \lrcorner\!\ulcorner \arrow[ur] \arrow[r] & 0 \end{tikzcd} \right)$ $\quad y = \left(\begin{tikzcd} \lrcorner\!\ulcorner \arrow[r] & 0 \\ 1 \arrow[ur] \arrow[r] & \lrcorner\!\ulcorner \end{tikzcd} \right)$ $\quad z = \left(\begin{tikzcd} 1 \arrow[r] & \lrcorner\!\ulcorner \\ 1 \arrow[ur] \arrow[r] & \lrcorner\!\ulcorner \end{tikzcd} \right)$
- $\delta_1^\nu x = \left(\begin{tikzcd} \lrcorner\!\ulcorner \arrow[r] & 0 \\ \nu \arrow[ur] \arrow[r] & 0 \end{tikzcd} \right)$ $\quad \delta_2^\nu x = \left(\begin{tikzcd} \nu \arrow[r] & 0 \\ \lrcorner\!\ulcorner \arrow[ur] \arrow[r] & 0 \end{tikzcd} \right)$ $\quad$ etc.



Properties

Proposition

For any interval order P there is a unique (up to …) track ρ in $\square^P$ with $\ell(\rho) = P$.

Proposition

Let X be a precubical set and P an interval order. There is a track ρ in X with $\ell(\rho) = P$ iff $\exists f : \square^P \to X$.

The construction $P \mapsto \square^P$ also works if P is not an interval order; but then the above do not hold.

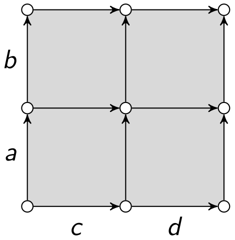
Interesting!?

The construction $P \mapsto \square^P$ also works if P is not an interval order.

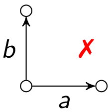
- For example for $P = \left(\begin{tikzcd} a \arrow[r] & b \\ c \arrow[r] \arrow[u, dotted] \arrow[ur, dotted] & d \arrow[u, dotted] \arrow[ul, dotted] \end{tikzcd}\right)$, $\square^P$ is

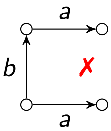
  (not a track object)

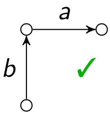
- (Generally, $\square^{P\|Q} \cong \square^P \otimes \square^Q$.)



So then, which precubical sets are poset objects?







- no “concave corners” / no “horns”
- connected Cat-0 sculptures?

Conclusion

Languages of HDA:

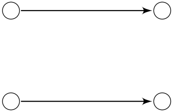
- subsumption-closed sets of interval orders ("weak")
- closed under union & parallel composition
- bisimulation invariant
- Towards a Kleene theorem for HDA!?

Precubical sets:

- new large precube category based on totally ordered sets
- very useful for us
- extensions: degeneracies $\Longleftarrow$ use non-injective maps
  symmetries $\Longleftarrow$ use unordered sets
- useful also for cubical HoTT? (see [Bezem-Coquand-Huber '13] for a similar category)

5 Posets for Concurrency: Interval Orders

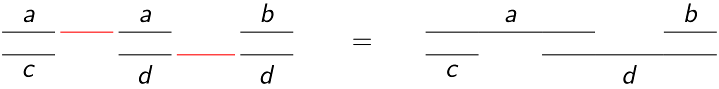
Posets for Concurrency: Interval Orders



- posets which are good for concurrency?
- already in [Wiener 1914], then [Winkowski '77], [Lamport '86], [van Glabbeek '90], [Vogler '91], [Janicky '93], etc.
- interval orders: posets which have representation as (real) intervals, ordered by $\max_1 \le \min_2$
- Lemma (Fishburn '70): A poset is interval iff it does not contain $\mathbf{II} = \left(\begin{smallmatrix} \cdot \longrightarrow \cdot \\ \cdot \longrightarrow \cdot \end{smallmatrix} \right)$ as induced subposet.
- intuitively: if $a \longrightarrow b$ and $c \longrightarrow d$, then also $a \longrightarrow d$ or $c \longrightarrow b$

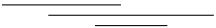
Gluing of Interval Orders

$$\begin{pmatrix} a \\ c \end{pmatrix} \overset{a}{*} \begin{pmatrix} a \\ d \end{pmatrix} \overset{d}{*} \begin{pmatrix} b \\ d \end{pmatrix} = \left(\begin{tikzcd} a \arrow[r] & b \\ c \arrow[ur] \arrow[r] & d \end{tikzcd}\right)$$



Interval Orders vs ST-Traces

- An ST-trace: $a^+ \, b^+ \, a^+ \, a^- \, a^- \, b^-$ [van Glabbeek '90]
- as intervals:



Proposition

ST-traces up to the equivalence generated by $a^+ b^+ \sim b^+ a^+$ and $a^- b^- \sim b^- a^-$ are the same as labeled interval orders.