A categorical framework for congruence of applicative bisimilarity in higher-order languages

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Based on previous work with Peio Borthelle.

Motivation: generalisation of theorem statements

- Often, theorems are stated for one “typical” programming language.
- Goal: provide high-level tools for stating them for all suitable languages and models.

State of the art

- Formats: Tyft/tyxt, GSOS, PATH,...
- Bialgebraic semantics (Turi and Plotkin '97).
  Functional languages only starting to be investigated (Peressotti '17).
- Reduction monads (Ahrens et al. '20).

Main contribution

1. Abstract setting for specifying operational semantics from signatures.
2. Abstract analogue of Abramsky's applicative bisimilarity in cbn λ-calculus, called substitution-closed bisimilarity.
3. A semantic format for congruence of substitution-closed bisimilarity:

Main theorem

If the signature complies with the format, then substitution-closed bisimilarity is a congruence.

Proof: abstract analogue of Howe's method.

This talk

Sketch main ideas on one example, big-step, cbn λ-calculus.

Call-by-name λ-calculus

Slightly non-standard presentation.

$$\frac{}{\lambda x.e \Downarrow e} \qquad\qquad \frac{e_1 \Downarrow e_3 \qquad e_3[e_2] \Downarrow e_4}{e_1\ e_2 \Downarrow e_4}$$

Typing: $\Downarrow \quad \subseteq \quad$ closed terms $\quad\times\quad$ terms with 1 free variable.

Syntactic graphs

Objects of interest:

- graphs with typed vertices,
- types $=$ natural numbers, morally numbers of potential free variables,
- sources are closed,
- targets have one potential free variable,
- vertices support the operations of untyped λ-calculus, including substitution.

Syntactic graphs

Definition

A syntactic graph consists of

- a model X_0 of syntax ($X_0(n)$ means n potential free variables), including

$$\lambda_n : X_0(n+1) \to X_0(n) \qquad app_n : X_0(n)^2 \to X_0(n)$$

$$subst_{p,n} : X_0(p) \times X_0(n)^p \to X_0(n),$$

- a set $X_{\Downarrow}$ of edges, and
- a source and target map $X_{\Downarrow} \to X_0(0) \times X_0(1)$.

They form a category $\Sigma_0\text{-}\mathbf{Gph}$.

Notation: $X_{\Downarrow} \to \Delta(X_0)$

Transition rules

A syntactic graph is a model of the rule

$$\frac{}{\lambda x.e \Downarrow e}$$

when it is equipped with

$$\begin{tikzcd} X_0(1) \arrow[rr, dashed] \arrow[dr, "e \mapsto (\lambda_1(e), e)"'] & & X_{\Downarrow} \arrow[dl] \\ & \Delta(X_0). & \end{tikzcd}$$

Intuition

For all potential parameters, there is a transition with expected source and target.

Transition rules

A syntactic graph is a model of the rule

$$\frac{\dfrac{r_1}{e_1 \Downarrow e_3} \qquad \dfrac{r_2}{e_3[e_2] \Downarrow e_4}}{e_1\ e_2 \Downarrow e_4}\ .$$

when it is equipped with

$$\begin{tikzcd}
A_\beta(X) \arrow[rr, dashed] \arrow[dr, "(r_1,e_2,r_2)\mapsto(app_0(s(r_1),e_2),t(r_2))"'] & & X_\Downarrow \arrow[dl] \\
& \Delta(X_0) &
\end{tikzcd}$$

where $A_\beta(X) = \{(r_1, e_2, r_2) \mid t(r_1)[e_2] = s(r_2)\}$.

Intuition

For all potential parameters, there is a transition with expected source and target.

Models

Definition

A model is a syntactic graph that is a model of both rules.

Informal Proposition

The initial model is a proof-relevant variant of the standard, syntactic graph.

Main result

Sketch: one defines

- substitution-closed relations,
- bisimulation,
- substitution-closed bisimilarity.

Theorem

Substitution-closed bisimilarity is a congruence.

Focus today: what makes the general theorem applicable to cbn λ.

Representable arities

Main steps:

- bisimulation by lifting and
- representable arities.

Bisimulation by lifting I

"Small" syntactic graphs:

- $\mathscr{L}(\mathbf{y}_0)$:
  - vertices generated from one closed constant, say k_0,
  - no transition.
- $\mathscr{L}(\mathbf{y}_{\Downarrow})$:
  - vertices generated from

$$k'_0 \in \mathscr{L}(\mathbf{y}_{\Downarrow})(0) \qquad \text{and} \qquad k'_1 \in \mathscr{L}(\mathbf{y}_{\Downarrow})(1),$$

  - one transition $r : k_0 \Downarrow k_1(x)$.

Proposition (≈ Yoneda)

- $\Sigma_0\text{-}\mathbf{Gph}(\mathscr{L}(\mathbf{y}_0), X) \cong X_0(0)$.
- $\Sigma_0\text{-}\mathbf{Gph}(\mathscr{L}(\mathbf{y}_{\Downarrow}), X) \cong X_{\Downarrow}$.
- $\mathscr{L}(\mathbf{y}_0) \xrightarrow{\mathscr{L}(\mathbf{y}_s)} \mathscr{L}(\mathbf{y}_{\Downarrow}) \xrightarrow{e} X$ corresponds to $s(e)$.

Bisimulation by lifting II

Definition

$f : X \to Y$ is a functional bisimulation iff

$$\begin{tikzcd}
\mathscr{L}(\mathbf{y}_0) \arrow[r, "x"] \arrow[d, "\mathbf{y}_s"'] & X \arrow[d, "f"] \\
\mathscr{L}(\mathbf{y}_\Downarrow) \arrow[r, "e'"'] \arrow[ur, dashed, "e"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r, mapsto] \arrow[d, Rightarrow, "e"'] & f(x) \arrow[d, Rightarrow, "e'"] \\
x' \arrow[r, mapsto] & y'.
\end{tikzcd}$$

(for all / exists)

Notation

$f \in \{\mathscr{L}(\mathbf{y}_s)\}^{\boxslash}$, generalises to $\mathbb{J}^{\boxslash}$.
$\mathscr{L}(\mathbf{y}_s) \in {}^{\boxslash}\{f\}$, generalises to ${}^{\boxslash}\mathbb{K}$.

Definition

- fibration: $\{\mathscr{L}(\mathbf{y}_s)\}^{\boxslash}$.
- cofibration: ${}^{\boxslash}(\{\mathscr{L}(\mathbf{y}_s)\}^{\boxslash})$.

Representable operation arities

By example: $e_1\ e_2$, head operation of $\dfrac{e_1 \Downarrow e_3 \qquad e_3[e_2] \Downarrow e_4}{e_1\ e_2 \Downarrow e_4}$.

Goal

Find E_{app} such that $X(0)^2 \cong \Sigma_0\text{-}\mathbf{Gph}(E_{app}, X)$, naturally in X.

Solution (merely saying that application has 2 arguments)

$E_{app} = \mathscr{L}(\mathbf{y}_0) + \mathscr{L}(\mathbf{y}_0)$.

By
$$\begin{aligned} X(0)^2 &\cong \Sigma_0\text{-}\mathbf{Gph}(\mathscr{L}(\mathbf{y}_0), X)^2 \\ &\cong \Sigma_0\text{-}\mathbf{Gph}(\mathscr{L}(\mathbf{y}_0) + \mathscr{L}(\mathbf{y}_0), X). \end{aligned}$$

Remark: $\mathscr{L}$ preserves coproducts, so $\mathscr{L}(\mathbf{y}_0) + \mathscr{L}(\mathbf{y}_0) \cong \mathscr{L}(\mathbf{y}_0 + \mathbf{y}_0)$, etc.

Representable rule arities

By example:
$$\dfrac{\dfrac{r_1}{e_1 \Downarrow e_3} \qquad \dfrac{r_2}{e_3[e_2] \Downarrow e_4}}{e_1\ e_2 \Downarrow e_4}\ .$$

Goal

Find E_β such that $A_\beta(X) \cong \Sigma_0\text{-}\mathbf{Gph}(E_\beta, X)$, naturally in X.

$$\begin{tikzcd}
\mathscr{L}(\mathbf{y}_0) \arrow[r, "\mathscr{L}(\mathbf{y}_s)"] \arrow[d, "t(k_\Downarrow)[k_0]"'] & \mathscr{L}(\mathbf{y}_\Downarrow) \arrow[d] \arrow[ddr, bend left, "r_2"] & \\
\mathscr{L}(\mathbf{y}_\Downarrow + \mathbf{y}_0) \arrow[r] \arrow[drr, bend right, "{[r_1,e_2]}"'] \arrow[ur, phantom, "\ulcorner", very near end] & E_\beta \arrow[dr, dashed] & \\
 & & X
\end{tikzcd}$$

Indeed, $A_\beta(X) \cong$ cocones to X
$\cong \Sigma_0\text{-}\mathbf{Gph}(E_\beta, X)$.

Semantic format

Proposition

Each rule yields a boundary morphism

$$\text{head operation arity} \quad \to \quad \text{rule arity}.$$

$$\begin{tikzcd}
 & \mathscr{L}(\mathbf{y}_0) \arrow[r] \arrow[d] & \mathscr{L}(\mathbf{y}_{\Downarrow}) \arrow[d] \\
\mathscr{L}(\mathbf{y}_0 + \mathbf{y}_0) \arrow[r] & \mathscr{L}(\mathbf{y}_{\Downarrow} + \mathbf{y}_0) \arrow[r] & E_\beta \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Condition

Each (head operation arity $\to$ rule arity) is a cofibration.

For $\dfrac{e_1 \Downarrow e_3 \qquad e_3[e_2] \Downarrow e_4}{e_1\ e_2 \Downarrow e_4}$: stability under pushouts and composition.

Conclusion

Semantic format for congruence of substitution-closed bisimilarity

Representable arities should form cofibrations.

- Shown here: example of cbn λ.
- In the paper: general framework + more examples.

Short-term future work

Languages with terms as Labels.