Directed homology theories based on algebras

work in progress

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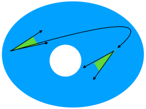
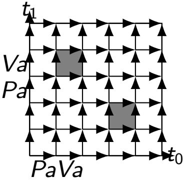
LHC, October 13th, 2022

Goal

Directed topology

Originated in problems in concurrency theory, and more generally in dynamical systems (e.g. differential inclusions): "characterize" topological spaces with a preferred direction (cone) of time, making paths in general non-invertible.





Ex. directed spaces [Grandis]

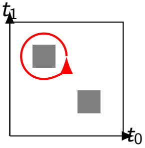
A directed space (Grandis) $\mathcal{X}$ consists of a pair (X, dX) where

- X is a topological space
- $dX \subseteq X^{[0,1]}$ is the set of directed paths
  - Every constant path is directed.
  - dX is closed under monotonic reparametrisation.
  - dX is closed under concatenation.

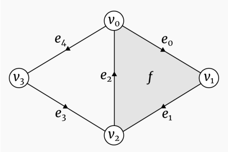
(but also, precubical sets, po-spaces, local po-spaces [Fajstrup-Goubault-Raussen], streams [Krishnan] etc.)

Classical homological invariants

E.g. first homology group (here, $\mathbb{Z}^2$)



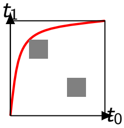
First generator (red cycle) Second generator Contractible cycle



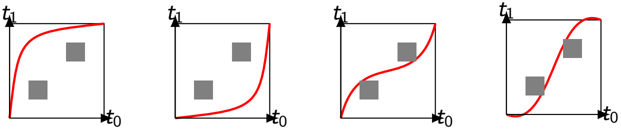
- Singular simplicial set from topological space
- Abelian chain complex from boundary operators
- Kernels (cycles) and images (boundaries): homology is defined as subquotients ("cycles modulo boundaries"), extremely well behaved in Abelian categories
- Hence, computation possible: convenient exact sequences or direct calculation

Goal

Dihomotopy/dihomeomorphism is finer than homotopy/homeomorphism



($Pa.Va.Pb.Vb$ | $Pb.Vb.Pa.Va$, 3 maximal dipaths mod dihom) Different from:

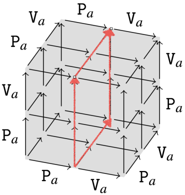


($Pa.Va.Pb.Vb$ | $Pa.Va.Pb.Vb$, 4 maximal dipaths mod dihom)

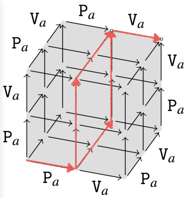
But, as topological spaces, they are homotopy equivalent (and $H_1 = \mathbb{Z}^2$).

Goal

Non NPC cubical complexes



2 non-dihomotopic dipaths which are homotopic



2 dihomotopic dipaths! (composition is not "regular")

Classical H_1 is trivial! ($H_2 = \mathbb{Z}$ though).

$\Longrightarrow$ homological invariant of some sort, with exact sequences etc.?
$\Longrightarrow$ Want to mimick the classical homological approach, as much as possible!

Some approaches in the directed case

Directed homology theories

Many attempts

- Categorical treatement, based on generalizations of bi-modules : natural systems à la Baues-Wirshing, see e.g. Jérémy Dubut, Eric Goubault, Jean Goubault-Larrecq: Directed Homology Theories and Eilenberg-Steenrod Axioms. Applied Categorical Structures 25(5), 2017 extension to natural systems with compositional pairing à la Porter, see e.g. Calk, Cameron & Goubault, E. & Malbos, Philippe. (2020). Time-reversal homotopical properties of concurrent systems. Homology, Homotopy and Applications. 22. 31-57. 10.4310/HHA.2020.v22.n2.a2.
- Patchkoria like approaches, in semi-modules (see also Sanjeevi Krishnan's work) - better categorical treatment from Alain Connes, Caterina Consani, "Homological Algebra in Characteristic One" would be applicable
- Factorization homology? Persistent homology? Stratified homology theories? Many leads under review (some for 30 years!)

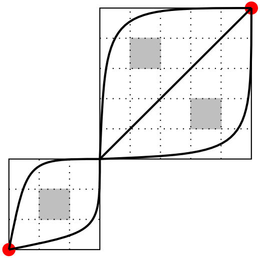
Pretty far from the "classical" approach

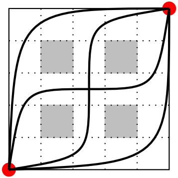
Will use algebras in this talk (see arxiv for modules over algebras, and current generalization on algebroids) that have the advantage of forming a semi-abelian category, not too far from the classical case!

Natural systems

Natural systems

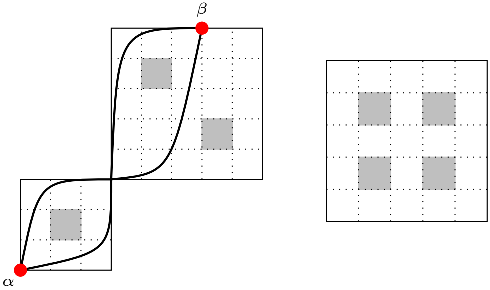
- Consider the topological space consisting of directed paths between points x and y.
- Apply classical algebraic invariants to these spaces.
- Allow the end-points x and y to vary [natural systems].





Non-dihomeomorphic directed spaces with homotopy-equivalent trace spaces between extremal points (eq. to 6 points).

Why natural systems?



Changing the base points distinguishes these directed spaces (eq. to 4 points on the left, eq. to only 1, 2, 3 or 6 points on the right).

Construction of these natural systems

Traces

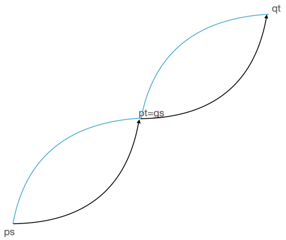
Let X be a directed space. A trace from p_s to p_t is a class of i-path modulo continuous increasing reparameterization.

Trace category

Define category $\boldsymbol{T}_1(X)$, $i = 1, \ldots$:

- $\boldsymbol{T}_1(X)$ has as objects, all points of X
- and as morphisms from s to t, all traces from s to t
- the composition is the obvious concatenation of such traces, which is associative because everything is taken modulo reparameterization:

$$p{*}q(s) = \begin{cases} p(2s) & \text{if } s \leq \frac{1}{2} \\ q(2s-1) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}$$



Natural homology

Factorization category of $\boldsymbol{T}_1(X)$

- Objects in $\mathcal{F}\boldsymbol{T}_1(X)$ are morphisms in $\boldsymbol{T}_1(X)$, i.e. dipaths modulo reparametrization
- Morphisms in $\mathcal{F}\boldsymbol{T}_1(X)$ from $p : x \to y$ to $q : x' \to y'$ are pairs of dipaths modulo reparametrization $\langle u, v\rangle$ s.t.:

$$\begin{tikzcd} x \arrow[r, "p"] & y \arrow[d, "v"] \\ x' \arrow[u, "u"] \arrow[r, "q"] & y' \end{tikzcd}$$

The natural homology of X is the collection of functors $HN_i(X) : \mathcal{F}\boldsymbol{T}_1(X) \to Ab$, $i \geq 1$, from the factorization category of $\boldsymbol{T}_1(X)$ to the category of abelian groups with:

$$HN_i(X)(p) = H_i(\boldsymbol{T}_1(X)(a, b))$$

where $p \in \boldsymbol{T}_1(X)$ is a 1-trace from a to $b \in X$, and H_i is the homology of $\boldsymbol{T}_1(X)(a, b)$,

$$HN_i(X)(\langle u, v\rangle)(p) = [v \circ p \circ u]$$

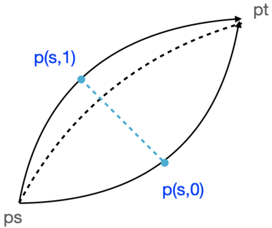
where $[x]$ denotes here the class in $H_i(X)(\boldsymbol{T}_1(X)(a', b')$ of x.

More generally: higher traces in directed spaces

Traces

Let X be a directed space, $i \geq 1$ an integer. We call p an i-path, or path of dimension i of X, any directed map

$$p : \boldsymbol{I} \times \Delta_{i-1} \to X$$



An i-trace from p_s to p_t is a class of i-path:

- modulo continuous increasing reparameterization in the first coordinate,
- with $p(0, t_0, \ldots, t_{i-1})$ not depending on $t_0, \ldots, t_{i-1}$ and equal to p_s
- and $p(1, t_0, \ldots, t_{i-1})$ constant as well, equal to p_t

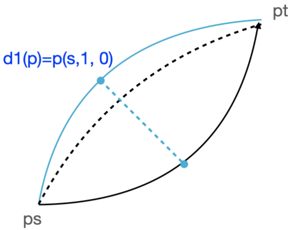
We write $T_i(X)$ for the set of i-traces in X. We write $T_i(X)(a, b)$, a, $b \in X$, for the subset of $T_i(X)$ made of i-traces from a to b.

Simplicial set structure on traces and higher traces

Boundaries and degeneracies

- Maps δ_j, $j = 0, \ldots, i$ acting on $(i+1)$-paths $p : \boldsymbol{I} \times \Delta_i \to X$, $i \geq 1$:

$$\delta_j(p) = p \circ (Id, d_j)$$



  with $d_j : \Delta_{i-1} \to \Delta_i$ is the standard inclusion map between standard simplexes

- Maps s_k, $k = 0, \ldots, i-1$ acting on i-paths $p : \boldsymbol{I} \times \Delta_{i-1} \to X$, $i \geq 1$:

$$\sigma_k(p) = p \circ (Id, s_k)$$

  where $s_k : \Delta_{i+1} \to \Delta_i$ is the usual surjective map between standard simplexes

Simplicial set structure on traces and higher traces

Simplicial set structure

The previous boundary and degeneracy operators give the sequence $ST(X) = (T_{i+1}(X))_{i\geq 0}$ the structure of a simplicial set.

Bi-grading

This restricts to a simplicial set $ST(X)(a,b)$ of traces from a to b in X, which is the singular simplicial set of the space of 1-traces (paths modulo continuous and increasing reparametrization) from a to b, with the compact-open topology.

Natural homology, revisited

$$HN_i(X)(p:\ a \to b) = H_{i-1}(ST(X)(a,b))$$

Trace categories

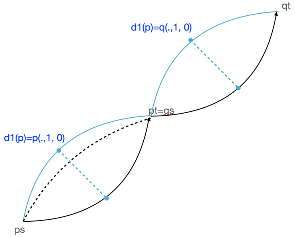
Trace and higher trace categories

Let X be a directed space, define categories $\boldsymbol{T}_i(X)$, $i = 1, \ldots$:

- $\boldsymbol{T}_i(X)$ has as objects, all points of X
- and as morphisms from s to t, all i-traces from s to t
- the composition is the obvious concatenation of such i-traces, which is associative because everything is taken modulo reparameterization:

$$p * q(s, t_0 \ldots, t_i) = \begin{cases} p(2s, t_0, \ldots, t_i) & \text{if } s \leq \frac{1}{2} \\ q(2s-1, t_0, \ldots, t_i) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}$$

This, plus previous observation: gives rise to a simplicial object in Cat!



We need coefficients now, that should not be Abelian!

Algebroids and algebras

Linearization

The linearization of $\boldsymbol{C}$ is the R-module enriched category which has as objects, the same objects as $\boldsymbol{C}$, and as morphisms from x to y in $\boldsymbol{C}$, the free R-module generated by $\boldsymbol{C}(x, y)$.
Note that linearization defines a functor from Cat, the category of categories, to $R - Mod - Cat$ the category of R-module enriched categories, or algebroids.

For now, one step further, by forgetting about the underlying objects of the linearization:

Category algebra

The category algebra or convolution algebra $R[\boldsymbol{C}]$ of $\boldsymbol{C}$ over R is the R-algebra whose underlying $R - module$ is the free module $R[\boldsymbol{C}_1]$ over the set of morphisms of $\boldsymbol{C}$ and whose product operation is defined on basis-elements $f, g \in \boldsymbol{C}_1 \subseteq R[\boldsymbol{C}]$ to be their composition if they are composable and zero otherwise:

$$f \times g = \begin{cases} g \circ f & \text{if composable} \\ 0 & \text{otherwise} \end{cases}$$

"Replace partial composition by product, with a 0 element".

The category of algebras

Algebras

An (non-unital) associative algebra on R, or R-algebra $A = (A, +, ., \times)$:

- is a R-module $(A, +, .)$ (with external multiplication by elements of the ring R denoted by .)
- that has an internal semigroup operation, which is an associative operation ("multiplication" or "internal multiplication")

$$\times : A \times A \to A$$

  that is bilinear.

We denote by 0 the neutral elements for $+$.

Morphisms of algebras

Let A and B be two R-algebras. A morphism $f : A \to B$ of R-algebras is a linear map from A to B seen as R-modules, such that it commutes with the internal multiplication:

$$f(a \times a') = f(a) \times f(a')$$

Algebras

Examples

- Polynomials (infinite dimensional, commutative): $R[t_1, \ldots, t_n]$ multivariate polynomials in n variables. Addition, multiplication by an element of R (scalar), multiplication of polynomials...
- Matrix algebras (finite dimensional, not commutative in general): sub-algebras of $\mathcal{M}_n(R)$ the algebra of $n \times n$ matrices, with addition, multiplication by an element of R (scalar) and matrix multiplication.

Examples of category algebras

On posets: incidence algebra (sub-algebra of the algebra of lower triangular matrices)

Consider for instance the following partial order:

$$1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n$$

Its path algebra is exactly the algebra of lower triangular matrices, that we denote by:

$$\begin{pmatrix} R & 0 & \dots & 0 & 0 \\ R & R & \dots & 0 & 0 \\ \dots & & & & \\ R & R & \dots & R & 0 \\ R & R & \dots & R & R \end{pmatrix}$$

Examples

On quivers (finite directed graphs): path algebra

Consider the simple loop:

$$\begin{tikzcd} 1 \arrow[loop right] \end{tikzcd}$$

- Its path algebra is easily seen to be the algebra of polynomials in one indeterminate $R[t]$
- Indeed, the basis of the R-module of dipaths is in bijection with $\{1, t, t^2, \ldots\}$ (where t^i denotes the unique path of length i),
- and the algebra multiplication adds up lengths of dipaths.

Examples

On quivers (finite directed graphs): path algebra

$$
\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & 2 \end{tikzcd}
$$

In that case, the corresponding path algebra is the following matrix algebra:

$$
\begin{pmatrix} R & 0 \\ R^2 & R \end{pmatrix}
$$

known as the Kronecker algebra. Elements of this Kronecker algebra are of the form:

$$
\begin{pmatrix} a & 0 \\ (b,c) & d \end{pmatrix}
$$

with internal (algebra) multiplication:

$$
\begin{pmatrix} a & 0 \\ (b,c) & d \end{pmatrix} \times \begin{pmatrix} a' & 0 \\ (b',c') & d' \end{pmatrix} = \begin{pmatrix} a'a & 0 \\ (a'b + b'd, a'c + c'd) & d'd \end{pmatrix}
$$

Of great importance in the representation theory of quivers and algebras, and in persistence algebra! (some other time)

Algebras from directed spaces

Higher traces algebras

For $i \geq 1$, we define the R-algebra $R_i[X]$ to be the category algebra of $\mathcal{T}_i(X)$.

But, careful!!

- The category algebra is not a functorial construction
- In fact, only injective-on-objects functors are mapped naturally onto algebra morphisms.

This will be a limitation for the practical use of corresponding exact sequences on directed spaces (solved by algebroids)

Simplicial algebras, out of directed spaces

Simplicial algebras are Kan!

- The collection of R-algebras $R[X] = (R_{i+1}[X])_{i \geq 0}$ can be given the structure of a Kan simplicial object in Alg.
- Either from [Van der Linden] or direct argument in the style of e.g. [May]: formulas for Horn filling already for a simplicial group structure.

And are bi-graded

Simplicial structure restricts to

$$R[X](a, b) = (R_{i+1}[X](a, b))_{i \geq 0}$$

which is the free (Kan) simplicial R-module on $ST(X)(a, b)$.
(could also be formalized as a bimodule of simplicial modules)

(could also be used for developing a directed homotopy theory - outside the scope of this presentation)

Chain complexes? But *Alg* is not abelian

Zero object in *Alg*

The zero object (in *Alg*, that is both initial and final object) is the associative algebra only containing 0. (Remember, we consider the category of non-unital algebras!)

Products and co-products in *Alg*

- Associative algebras are models of an algebraic theory in the sense of Lawvere, making easy to show completeness/co-completeness
- Products are "obvious" (underlying set is the product of the underlying sets); coproduct is much more involved (hence *Alg* is not even additive, so definitely not Abelian)

Semi-abelian categories

Interest of semi-abelian categories

- Enough "control" on the way sub-objects and quotients behave, so that "we get all exact sequences we are used to" in the classical (Abelian) case
- Extends Abelian categories (prototypal example being the category of Abelian groups) to e.g. groups, and also, algebras

One of the many characterizations: $\mathcal{C}$ is semi-abelian if...

- it has finite products and coproducts and a zero object;
- it has pullbacks of monomorphisms
- it has coequalizers of kernel pairs;
- regular epimorphisms are stable under pullback;
- equivalence relations are effective
- and the Split Short Five Lemma holds:

Going to skim through the red parts, with examples in the category of groups versus abelian groups.

Interest of semi-abelian homology

Exact sequences

For homology theories defined from simplicial objects in a semi-abelian category, any short exact sequence in the category of simplicial objects:

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

gives rise to a corresponding long exact sequence in homology [Van der Linden]:

$$\begin{tikzcd}
H_c^{q-1}(C) & H_c^{q-1}(B) \arrow[l] & H_c^{q-1}(A) \arrow[l] \\
H_c^{q}(C) \arrow[urr, "d^*"] & H_c^{q}(B) \arrow[l] & H_c^{q}(A) \arrow[l]
\end{tikzcd}$$

(e.g. Mayer-Vietoris ["how to compute the homology of a union of spaces"] etc.)

Regular morphisms - generalized subobjects and quotients

Definition

- Regular (resp. normal) monomorphism p: equalizer of a parallel pair of morphisms:

$$\begin{tikzcd} P \arrow[r, "p"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\ P' \arrow[u, dotted] \arrow[ur, "p'"'] & & \end{tikzcd}$$

  (resp. with $g = 0$)
  Notations: $ker(f,g): \; Ker(f,g) \rightarrow X$ (resp. $ker(f): \; Ker(f) \rightarrow X$)

- Regular (resp. normal) epimorphism q ("generalized quotient maps"): dual of regular monomorphism, a coequalizer of a parallel pair of morphisms:

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "q"] \arrow[dr, "q'"'] & Q \arrow[d, dotted] \\ & & Q' \end{tikzcd}$$

  Notations, *coKer* and *coker*.

Regular morphisms: groups vs abelian groups

Regular and normal monos

- In *Ab*, all inclusions of subgroups are normal, hence regular.
- In *Grp*,
  - the monomorphisms are inclusions of subgroups, and every monomorphism is regular.
  - but not all monos are normal (only inclusions of normal subgroups $N \rightarrow G$; i.e. $gNg^{-1} \in G$ for all $g \in G$)

Regular and normal epis

- In *Grp*, every epimorphism is normal, hence regular (is surjective!)
- Same in *Ab*

Regular category

For a (finitely) complete and co-complete category

The pullback of a regular epimorphism along any morphism is again a regular epimorphism

Consequence: nice properties for the "image" of a morphism and subquotients

- In a regular category, the regular-epimorphisms and the monomorphisms form a factorization system: $f: X \rightarrow Y$ can be factorized uniquely into a regular epimorphism $e: X \rightarrow E$ followed by a monomorphism $m: E \rightarrow Y$,
- The monomorphism m is called the image of f.

The case of *Grp* and *Ab*

- *Grp* and *Ab* are regular: we always have a factorization as injective function composed with surjective function, as in *Set*!
- In more details: the pullback of the regular epimorphism $f: G \rightarrow K$ along $f': H \rightarrow K$ is
$$P = \{(g, h) \in G \times H \mid f(g) = f'(h)\}$$
with the canonical projections. Then $\pi_2: P \rightarrow H$ is a (regular) epimorphism.

Barr-exactness

Barr-exactness=regular+all equivalence relations are effective ("defined by a kernel pair!")

Equivalence relations in finitely complete categories

A relation between objects X and Y in a category is a span of maps $d_1 : R \to X$ and $d_2 : R \to Y$ such that the induced map $R \to X \times Y$ is a mono. "Obvious" categorical formalization of reflexivity, symmetry and transitivity.

Kernel pair

For $f : X \to Y$, its kernel pair is the relation R between X and X obtained by pullback between f and itself:

$$\begin{tikzcd} R \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Kernel pairs are always equivalence relations in finitely complete categories. Converse?

Effective equivalence relations, and groups

Effective equivalence relations

An equivalence relation is effective if it is a kernel pair.

In groups

- a relation R is a mono (subobject of $X \times X$) $r : R \to X \times X$: let S be the congruence class of the neutral element $1 \in X$:

$$\{xyx^{-1} \mid x \in X,\ (1,y) \in R\}$$

- S is a normal subgroup of X
- R is the kernel pair of the canonical quotient map $f : X \to X/S$

Short Five Lemma

Short Five Lemma

$$\begin{tikzcd}
Ker\ f \arrow[r, "ker\ f"] \arrow[d, "k"'] & A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[d, "b"] \\
Ker\ f' \arrow[r, "ker\ f'"'] & A' \arrow[r, "f'"'] & B'
\end{tikzcd}$$

Suppose that in the diagram above, f and f' are regular epimorphisms and k and b are isomorphisms, we have to show that a is an isomorphism.

In the category of groups

- We can prove that a is injective and surjective
- By a direct "elementwise" argument.

Alg is semi-abelian

In a nutshell, the "fast-track" argument

- *Alg* forms an algebraic variety in the sense of Lawvere,
- Algebraic varieties are semi-abelian when e.g. they have a zero object and are protomodular (in the sense of Bourn - see e.g. [Van der Linden])
- By algebraic reasoning, an algebraic variety is protomodular (one of the other many characterizations of semi-abelian categories) in particular when the algebraic variety includes a group operation (see e.g. [Borceux]), which is obviously the case for *Alg*.

(this is why we exemplified the case of *Grp*, this is enough for understanding these concepts)

A bit more in details: about subobjects in Alg

Equalizers

- The equalizer of $f,\ g:\ A \to B$ is

$$Ker(f,g) = \{a \in A \mid f(a) = g(a)\}$$

  with the inclusion morphism $ker(f,g):\ Ker(f,g) \to A$.
- We write $Ker\ f$ for $Ker(f,0)$ and $ker\ f:\ Ker\ f \to A$
- We call $Ker(f)$, $Ker(f,0)$, the kernel of f.

Normal monomorphisms

A normal mono is the equalizer of 0 and some morphism (i.e. the kernel of a morphism). It is easy to see that they correspond to inclusions of two-sided ideals within an algebra A.

Another important kind of morphisms: proper morphisms

Proper morphisms are morphisms whose image is normal (i.e. the image is a two-sided ideal in Alg).

Principle of our approach

Reminder: $R[X] = (R_i[X])_{i \geq 0}$ is a (Kan) simplicial object in Alg. What do we do from here?

Properness and Moore normalization

- Define naively:
$$H_n = Ker\ \partial / I(\partial)$$
where $I(\partial)$ is the two-sided ideal generated by $Im\ \partial$ (∂ is not proper in general) - will not work for "controlling" exact sequences
- Moore normalization of a simplicial object (here in the semi-abelian category Alg) is a proper chain complex: the boundary map is proper.
- This has no reason to give the same "homology algebras" (as would be the case in e.g. simplicial groups) - we characterize them as well

Moore normalization

Of simplicial algebras

Let S be a simplicial object in Alg. Its Moore normalization is the following chain complex of algebras, NS:

- is in degree n the joint kernel $(NS)_n = \bigcap_{i=1}^{n} ker\ d_i^n$ of all face maps except the 0-face;
- with differential given by the remaining 0-face map $\partial^n = d_0^n : \ (NS)_n \to (NS)_{n-1}$

Homology algebras

The homology algebra of a directed space X is defined as the the homology $(HA_{i+1}(X))_{i \geq 0}$ in the semi-abelian category Alg of the simplicial object, shifted by one, i.e. for all $i \geq 0$:

$$HA_{i+1}(X) = H_i(NR[X])$$

i.e. is the classical homology of the normalized simplicial algebra $NR[X]$.

(leaving out for now the construction of $HA_0(X)$ which is quite distinct)

Properties of homology algebras

First homology algebra

$HA_1(X)$ is the coequalizer of δ_0 and δ_1 from $R_2[X]$ to $R_1[X]$. More precisely, this is:

$$HA_1(X) = R_1[X]/\sim$$

where p and $q \in R_1[X]$ are such that $p \sim q$ if and only if there exists $z \in R_2[X]$ with:

$$p = q + (\delta_0 - \delta_1)(z)$$

This is definitely what we expect: generated by directed paths modulo "transversal" deformation (dihomotopy), with algebra operation being concatenation

Examples

(ignoring here the precise definition in the precubical case and the correspondance with the homology of the geometric realization as a directed space)

Filled-in square

$$
\begin{tikzcd}
4 \arrow[r, "a"] \arrow[d, "b"] \arrow[dr, phantom, "C"] & 2 \arrow[d, "c"] \\
3 \arrow[r, "d"] & 1
\end{tikzcd}
$$

The path algebra $R_1[X]$

$$
\begin{pmatrix}
R & 0 & 0 & 0 \\
R & R & 0 & 0 \\
R & 0 & R & 0 \\
R^2 & R & R & R
\end{pmatrix}
$$

(remember directed S^1 and the Kronecker algebra).

Examples

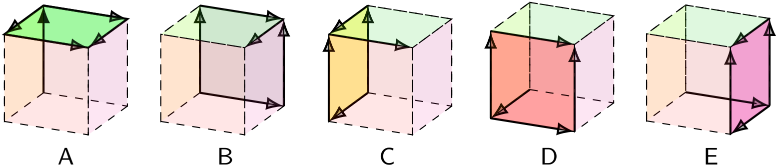
$HA_1(X)$

Quotient $R_1[X]$ by the (admissible, two-sided) ideal generated by $ac - bd$ ($= Im\ \partial$), can be seen as the matrix algebra:

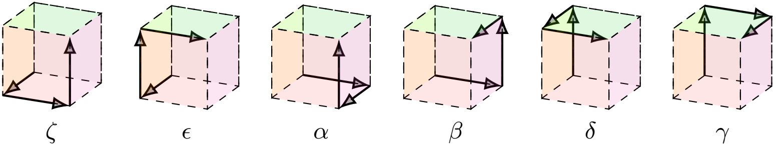
$$\begin{pmatrix} R & 0 & 0 & 0 \\ R & R & 0 & 0 \\ R & 0 & R & 0 \\ R & R & R & R \end{pmatrix}$$

Example: Fahrenberg's matchbox

2-traces "generated" by:



1-traces are "generated" by:



Examples

Fahrenberg's matchbox: path algebra and $HA_1(X)$

$$\begin{pmatrix} R & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ R & R & 0 & 0 & 0 & 0 & 0 & 0 \\ R & 0 & R & 0 & 0 & 0 & 0 & 0 \\ R^2 & R & R & R & 0 & 0 & 0 & 0 \\ R & 0 & 0 & 0 & R & 0 & 0 & 0 \\ R^2 & R & 0 & 0 & R & R & 0 & 0 \\ R^2 & 0 & R & 0 & R & 0 & R & 0 \\ R^6 & R^2 & R^2 & R & R^2 & R & R & R \end{pmatrix}$$

Path algebra $R_1[X]$

$$\begin{pmatrix} R & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ R & R & 0 & 0 & 0 & 0 & 0 & 0 \\ R & 0 & R & 0 & 0 & 0 & 0 & 0 \\ R & R & R & R & 0 & 0 & 0 & 0 \\ R & 0 & 0 & 0 & R & 0 & 0 & 0 \\ R & R & 0 & 0 & R & R & 0 & 0 \\ R & 0 & R & 0 & R & 0 & R & 0 \\ R & R & R & R & R^2 & R & R & R \end{pmatrix}$$

$H_1[X] = R_1[X]/Im\ \partial$, hence is the bound quiver algebra $(X_{\leq 1}, Im\ \partial)$

Higher algebras

Underlying module structure

Let X be a directed space, $a, b \in X$, R a ring. Then $HA_n[X](a, b) = e_a^n HA_n[X] e_b^n$, $n \geq 1$, is the standard nth homology (taken in the abelian category of R-modules) of the trace space from a to b, that is,

$$HA_n[X](a, b) = HN_n(X)(p)$$

as a R-module, with p any 1-trace from a to b.

Algebra structure

From [Van der Linden], for all $n \geq 2$, $HA_n(X)$ is an abelian object in Alg hence is just a R-module! $(\times = +)$

Abelian objects in *Alg*

Abelian objects

A is an abelian object in category $\mathcal{C}$ is there exists a (necessarily unique) morphism: $m : \ A \times A \to A$ such that:

$$\begin{tikzcd} A \arrow[d, "(Id_A{,}0)"'] \arrow[dr, "="] & \\ A \times A \arrow[r, "m"] & A \\ A \arrow[u, "(0{,}Id_A)"] \arrow[ur, "="'] & \end{tikzcd}$$

In semi-abelian categories, they are even internal abelian groups.

Abelian objects in *Alg*

We thus have three monoid operations in any abelian objects in Alg, m, $+$, $\times$. By (twice) the Eckmann-Hilton argument:

- $m = + = \times$ (which is commutative)
- with the same neutral elements

Hence abelian objects in Alg are just modules ($\times = +$).

Example

Empty cube

Consider the cubical complex whose underlying quiver is:

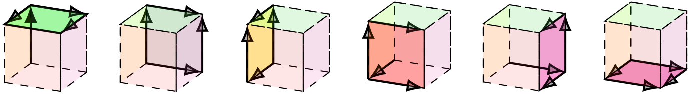
$$\begin{tikzcd}
 & 4 \arrow[rr] \arrow[dl] & & 3 \arrow[dl] \\
2 \arrow[rr] & & 1 & \\
 & 8 \arrow[uu] \arrow[rr] \arrow[dl] & & 7 \arrow[uu] \arrow[dl] \\
6 \arrow[uu] \arrow[rr] & & 5 \arrow[uu] &
\end{tikzcd}$$

and whose 6 faces $(8, 6, 5, 7)$, $(6, 5, 2, 1)$ etc. are filled in by 2-cells.

HA_2(empty cube)

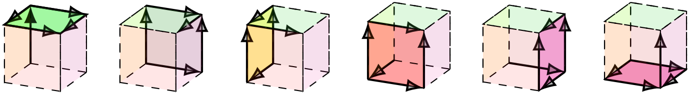
$R_2[X]$ generated...

...by the following 2-paths of length 1:



A' B' C' D' E' F'

Plus by their whiskering which are 2-paths of length 2:



A B C D E F

$H_2(X)$ is (the R-module) R, generated by $A + B + C + D + E + F$.

There is more!

Links with natural systems with compositional pairing

See see e.g. Calk, Cameron & Goubault, E. & Malbos, Philippe. (2020). Time-reversal homotopical properties of concurrent systems. Homology, Homotopy and Applications. 22. 31-57. 10.4310/HHA.2020.v22.n2.a2. and the (forthcoming) arxiv paper.

Practical computations

Using GAP

Homology via modules over the algebra $R_1[X]$

- Link with representation theory of quivers and algebras,
- And persistence theory ("over the trace category")

See the arxiv paper and forthcoming paper with Cameron Calk and Philippe Malbos.

From algebras to algebroids

- Ongoing work
- Implies functoriality from directed spaces
- At the expense of having to do most of the work "by hand" (quasi-pointed category, no direct semi-abelian technique available)