A functorial excursion
between linear logic and algebraic geometry

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Starting point and motivating analogy

In algebraic geometry, there are two kinds of spaces:

- ▷ **schemes** which may be seen as **commutative rings** dualized into **affine schemes** and "glued together" in an appropriate way,
- ▷ **bundles** usually described as **quasi-coherent modules** over the **structure sheaf of rings** a specific scheme X.

Much progress has been made to design **sheaf models** of dependent and homotopy type theory. There, a type is interpreted as a **sheaf** or a **space**.

The position of linear logic is not entirely clear from that point of view. Could we understand linear logic as a **logic of bundles** on spaces?

The category $\mathbf{Mod}_R$ of modules

Every **symm. monoidal closed category** defines a model of linear logic.

Hence: the category $\mathbf{Mod}_R$ of R-modules for a given commutative ring R.

Conjunction as tensor product:

$M \otimes_R N$ as the abelian group $M \otimes N$ quotiented

Implication and hypothetical reasoning as linear hom:

$M \multimap_R N$ as the abelian group of R-module homomorphisms.

Purpose of this talk: extend / adapt this interpretation to **presheaves of modules** over a **covariant presheaf** $X \in [\mathbf{Ring}, \mathbf{Set}]$ of commutative rings.

An axiomatic approach to abelian groups

We want to axiomatize the properties of the category $\mathscr{A} = \mathbf{Ab}$ of abelian groups and homomorphisms between them.

We suppose given a **symmetric monoidal category**

$$(\mathscr{A}, \otimes, 1)$$

where every reflexive pair

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

has a **coequalizer**, preserved by the tensor product on each component.

Reflexive pairs

A **reflexive pair** in a category $\mathscr{A}$ is a pair of maps

$$
\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}
$$

such that there exists a common **section** of the two maps f and g

$$
\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, shift right=2, "g"'] & B \arrow[l, "s" description] \end{tikzcd}
$$

in the sense that the equations hold:

$$
f \circ s \quad = \quad id_B \quad = \quad g \circ s
$$

Rings as commutative monoid objects

A **commutative ring** is an object $R \in \mathscr{A}$ equipped with two maps

$$m : R \otimes R \to R \qquad\qquad e : 1 \to R$$

making the diagrams commute:

$$\begin{tikzcd}
R \otimes R \otimes R \arrow[r, "m\otimes R"] \arrow[d, "R\otimes m"'] & R \otimes R \arrow[d, "m"] \\
R \otimes R \arrow[r, "m"] & R
\end{tikzcd}
\qquad
\begin{tikzcd}
& R \otimes R \arrow[dr, "m"] & \\
R \arrow[ur, "e\otimes R"] \arrow[rr, "id_R"] \arrow[dr, "R\otimes e"'] & & R \\
& R \otimes R \arrow[ur, "m"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
R \otimes R \arrow[rr, "\gamma_{R,R}"] \arrow[dr, "m"'] & & R \otimes R \arrow[dl, "m"] \\
& R &
\end{tikzcd}$$

The category $\mathbf{Ring}$ of commutative rings

Given two rings R and S, a **ring homomorphism**

$$u \quad : \quad (R, m_R, e_R) \longrightarrow (S, m_S, e_S)$$

is a map of the category $\mathscr{A}$

$$u \quad : \quad R \longrightarrow S$$

making the diagrams commute:

$$\begin{tikzcd}
R \otimes R \arrow[r, "u \otimes u"] \arrow[d, "m_R"'] & S \otimes S \arrow[d, "m_S"] \\
R \arrow[r, "u"] & S
\end{tikzcd}
\qquad
\begin{tikzcd}
& 1 \arrow[dl, "e_R"'] \arrow[dr, "e_S"] & \\
R \arrow[rr, "u"] & & S
\end{tikzcd}$$

The category $\mathbf{Ring}$ of commutative rings

The category $\mathbf{Ring}$ is defined as the category

- ▷ whose objects are the **commutative rings** of the category $\mathscr{A}$,
- ▷ whose maps are the **ring homomorphisms** between them.

Note that the category $\mathbf{Ring}$ has **finite sums** defined by the tensor product.

The sum of two commutative rings R and S is the commutative ring $R \otimes S$ with multiplication map defined using the symmetry:

$$R \otimes S \otimes R \otimes S \xrightarrow{R \otimes \gamma_{R,S} \otimes S} R \otimes R \otimes S \otimes S \xrightarrow{m_R \otimes m_S} R \otimes S$$

and terminal object the monoidal unit 1 seen as a commutative ring.

The category $\mathbf{Mod}_R$ of modules over a ring R

Suppose given a commutative ring R.

An R-**module** is an object $M \in \mathscr{A}$ equipped with a map

$$\mathrm{act} \quad : \quad R \otimes M \longrightarrow M$$

making the diagrams below commute:

$$\begin{tikzcd} R \otimes R \otimes M \arrow[r, "m_R \otimes M"] \arrow[d, "R \otimes \mathrm{act}"'] & R \otimes M \arrow[d, "\mathrm{act}"] \\ R \otimes M \arrow[r, "\mathrm{act}"] & M \end{tikzcd} \qquad \begin{tikzcd} & R \otimes M \arrow[dr, "\mathrm{act}"] & \\ M \arrow[ur, "e_R \otimes M"] \arrow[rr, "id_M"] & & M \end{tikzcd}$$

Equivalently, an R-module is an **Eilenberg-Moore algebra** for the monad

$$A \mapsto R \otimes A \quad : \quad \mathscr{A} \longrightarrow \mathscr{A}$$

induced by the commutative ring R in the category $\mathscr{A}$.

The category $\mathbf{Mod}_R$ of modules over a ring R

A R**-module homomorphism** between R-modules

$$f \quad : \quad (M, \mathrm{act}_M) \longrightarrow (N, \mathrm{act}_N)$$

is a map $f : M \to N$ making the diagram commute:

$$\begin{tikzcd} R \otimes M \arrow[r, "R\otimes f"] \arrow[d, "\mathrm{act}_M"'] & R \otimes N \arrow[d, "\mathrm{act}_N"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

We write $\mathbf{Mod}_R$ for the category:

- ▷ whose objects are the R**-modules**,
- ▷ whose maps are the R**-module homomorphisms** between them.

The category $\mathbf{Mod}$ of modules

A **module** is a pair (R, M) consisting of

- ▷ a commutative ring R
- ▷ an R-module (M, act_M)

A **module homomorphism**

$$(u, f) \quad : \quad (R, M) \to (S, N)$$

is a pair consisting of

- ▷ a ring homomorphism $u : R \to S$
- ▷ a map $f : M \to N$ making the diagram commute:

$$\begin{tikzcd} R \otimes M \arrow[r, "u \otimes f"] \arrow[d, "\mathrm{act}_M"'] & S \otimes N \arrow[d, "\mathrm{act}_N"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

The category $\mathbf{Mod}$ of modules

The category $\mathbf{Mod}$ is defined as the category

- ▷ whose objects are the **modules**,
- ▷ whose maps are the **module homomorphisms** between them.

There is an obvious functor

$$\pi \quad : \quad \mathbf{Mod} \longrightarrow \mathbf{Ring}$$

which transports every module (R, M) to its underlying commutative ring R.

For that reason, we find convenient to write

$$u : R \longrightarrow S \quad \models \quad f : M \longrightarrow N$$

for a module homomorphism $(u, f) : (R, M) \to (S, N)$.

The category $\mathbf{Mod}$ of modules

The notation

$$u \ : \ R \longrightarrow S \quad \models \quad f \ : \ M \longrightarrow N$$

is inspired by the intuition that every ring homomorphism

$$u \quad : \quad R \longrightarrow S$$

induces a **fiber** consisting of all the module homomorphisms of the form

$$(u, f) \quad : \quad (R, M) \longrightarrow (S, N)$$

equivalently, of all the maps $f : M \to N$ making the diagram commute:

$$\begin{tikzcd} R \otimes M \arrow[r, "u \otimes f"] \arrow[d, "\mathrm{act}_M"'] & S \otimes N \arrow[d, "\mathrm{act}_N"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

Note that $\mathbf{Mod}_R$ is the fiber of the identity map $id_R \ : \ R \to R$.

The Grothendieck bifibration $\pi : \mathbf{Mod} \to \mathbf{Ring}$

A well-known fact is that the functor

$$\pi \quad : \quad \mathbf{Mod} \longrightarrow \mathbf{Ring}$$

defines a Grothendieck bifibration.

Every ring homorphism

$$u \quad : \quad R \longrightarrow S$$

induces a **restriction/extension adjunction** between the fiber categories:

$$\begin{tikzcd} \mathbf{Mod}_R \arrow[r, shift left, "\mathbf{ext}_u"] \arrow[r, phantom, "\perp"] & \mathbf{Mod}_S \arrow[l, shift left, "\mathbf{res}_u"] \end{tikzcd}$$

The restriction of scalar functor

Every S-module (N, act_N) induces a R-module noted

$$\mathbf{res}_u\, N \quad = \quad (N, \mathrm{act}'_N)$$

with same underlying object N as the original S-module, and with action

$$\mathrm{act}'_N : R \otimes N \to N$$

defined as the composite:

$$\mathrm{act}'_N \quad = \quad R \otimes N \xrightarrow{u \otimes N} S \otimes N \xrightarrow{\mathrm{act}_N} N$$

The S-module (N, act_N) comes moreover with a module homomorphism

$$u \ : \ R \longrightarrow S \quad \models \quad id_N \ : \ \mathbf{res}_u\, N \longrightarrow N \tag{1}$$

which is cartesian in the (original) sense of Grothendieck.

The extension of scalar functor

The restriction of scalar functor

$$\mathbf{res}_u \quad : \quad \mathbf{Mod}_S \longrightarrow \mathbf{Mod}_R$$

has a **left adjoint** noted

$$\mathbf{ext}_u \quad : \quad \mathbf{Mod}_R \longrightarrow \mathbf{Mod}_S$$

One way to construct the functor $\mathbf{ext}_u$ is to define the $R \otimes S$-module

$$R \otimes_u S$$

as the **reflexive coequalizer** of the diagram:

$$\begin{tikzcd} R \otimes R \otimes S \arrow[r, shift left=2, "m_R \otimes S"] \arrow[r, shift right=2, "(R \otimes m_S) \circ (R \otimes u \otimes S)"'] & R \otimes S \arrow[l, "R \otimes e_R \otimes S" description] \end{tikzcd}$$

The extension of scalar functor

Given three rings R, S_1 and S_2, we define the **composition functor**

$$\circledast_R \ : \ \mathbf{Mod}_{S_1 \otimes R} \times \mathbf{Mod}_{R \otimes S_2} \longrightarrow \mathbf{Mod}_{S_1 \otimes S_2}$$

which transports a pair (M, N) consisting of $\begin{cases} \text{a } S_1 \otimes R\text{-module } M \\ \text{a } R \otimes S_2\text{-module } N \end{cases}$

to the $S_1 \otimes S_2$-module $M \otimes_R N$ defined as the reflexive coequalizer of

$$\begin{tikzcd} M \otimes R \otimes N \arrow[r, shift left=2, "\mathrm{act}_M \otimes N"] \arrow[r, shift right=2, "M \otimes \mathrm{act}_N"'] & M \otimes N \arrow[l, "M \otimes e_R \otimes N"' description] \end{tikzcd}$$

Here, the two maps $\mathrm{act}_M : M \otimes R \to M$ and $\mathrm{act}_N : R \otimes N \to N$ are deduced from the $S_1 \otimes R$-module structure of M and $R \otimes S_2$-module structure of N, by restriction of scalar along $R \to S_1 \otimes R$ and $R \to R \otimes S_2$.

The extension of scalar functor

The left adjoint functor

$$\mathbf{ext}_u \quad : \quad \mathbf{Mod}_R \longrightarrow \mathbf{Mod}_S$$

is defined as

$$\mathbf{ext}_u \quad : \quad M \quad \mapsto \quad M \circledast_R (R \otimes_u S)$$

by applying the $R \otimes S$-module

$$R \otimes_u S$$

on the R-module M using the composition functor

$$\circledast_R \ : \ \mathbf{Mod}_R \times \mathbf{Mod}_{R \otimes S} \longrightarrow \mathbf{Mod}_S$$

An axiomatic approach to abelian groups (2)

From now on, we make the extra assumption that

the category $\mathscr{A}$ is **symmetric monoidal closed**

and has all coreflexive equalizers.

The internal hom-object in $\mathscr{A}$ is noted $\mathbf{Hom}(M, N)$.

The category $\mathbf{Mod}^{\ominus}$ of modules and retromorphisms

A **module retromorphism**

$$(u, f) \quad : \quad (R, M) \to (S, N)$$

is a pair consisting of

- a ring homomorphism $u : R \to S$
- a map $f : N \to M$ making the diagram commute:

$$\begin{tikzcd}
R \otimes M \arrow[d, "\mathrm{act}_M"'] & R \otimes N \arrow[l, "R \otimes f"'] \arrow[r, "u \otimes N"] & S \otimes N \arrow[d, "\mathrm{act}_N"] \\
M & & N \arrow[ll, "f"']
\end{tikzcd}$$

The category $\mathbf{Mod}^{\ominus}$ of modules and retromorphisms

The category $\mathbf{Mod}^{\ominus}$ is defined as the category

- whose objects are the **modules**,
- whose maps are the **module retromorphisms** between them.

There is an obvious functor

$$\pi^{\ominus} \quad : \quad \mathbf{Mod}^{\ominus} \longrightarrow \mathbf{Ring}$$

which transports every module (R, M) to its underlying commutative ring R.

Note that the functor $\pi^{\ominus}$ is a Grothendieck fibration, which coincides in fact with the **opposite** of the Grothendieck fibration π.

The Grothendieck bifibration $\pi^{\ominus} : \mathbf{Mod}^{\ominus} \to \mathbf{Ring}$

It turns out that the functor

$$\pi^{\ominus} \quad : \quad \mathbf{Mod}^{\ominus} \longrightarrow \mathbf{Ring}$$

defines in fact a Grothendieck bifibration.

The reason is that every ring homorphism

$$u \quad : \quad R \longrightarrow S$$

induces a **restriction/coextension adjunction** between fiber categories:

$$\begin{tikzcd} \mathbf{Mod}^{\ominus}_{R} \arrow[r, shift left=2, "\mathbf{coext}_{u}"] \arrow[r, phantom, "\perp"] & \mathbf{Mod}^{\ominus}_{S} \arrow[l, shift left=2, "\mathbf{res}_{u}"] \end{tikzcd}$$

where the category $\mathbf{Mod}^{\ominus}_{R}$ is the opposite of the category $\mathbf{Mod}_{R}$.

The coextension of scalar functor

The restriction of scalar functor

$$\mathbf{res}_{\,u} \quad : \quad \mathbf{Mod}_S^{\ominus} \longrightarrow \mathbf{Mod}_R^{\ominus}$$

has a **left adjoint** noted

$$\mathbf{coext}_{\,u} \quad : \quad \mathbf{Mod}_R^{\ominus} \longrightarrow \mathbf{Mod}_S^{\ominus}$$

The functor $\mathbf{coext}_u$ transports every R-module (M, act_M) to the S-module

$$\mathbf{coext}_{\,u}(M) \quad = \quad [S, M]_u$$

defined as the coreflexive equalizer of the diagram:

$$\begin{tikzcd}[column sep=huge]
\mathbf{Hom}(S,M) \arrow[r, shift left=2, "\mathbf{Hom}(u\otimes S\,,M)\circ\mathbf{Hom}(\,m_S\,,M)"] \arrow[r, shift right=2, "\mathbf{Hom}(R\otimes S\,,\mathrm{act}_M)\circ\mathbf{Hom}(R\otimes -, R\otimes -)"'] & \mathbf{Hom}(R\otimes S, M) \arrow[l, "\mathbf{Hom}(e_R\otimes S\,,M)" description]
\end{tikzcd}$$

The coextension of scalar functor

The coreflexive equalizer $\mathbf{coext}_u(M)$ provides an internal description in the category $\mathscr{A}$ of the set of maps $f : S \to M$ making the diagram commute:

$$\begin{tikzcd}
R \otimes M \arrow[dd, "\mathrm{act}_M"'] & & R \otimes S \arrow[ll, "R\otimes f"'] \arrow[d, "u\otimes S"] \\
& & S \otimes S \arrow[d, "m_S"] \\
M & & S \arrow[ll, "f"']
\end{tikzcd}$$

or equivalently, as the set of R-module homomorphisms $f : \mathbf{res}_u S \to M$.

The trifibration $\pi : \mathbf{Mod} \to \mathbf{Ring}$ of modules

Putting together all the constructions, every ring homomorphism

$$R \xrightarrow{\quad u \quad} S$$

induces three functors

$$\begin{tikzcd} \mathbf{Mod}_R \arrow[r, shift left=4, "\mathbf{coext}\,u"] \arrow[r, shift right=4, "\mathbf{ext}\,u"'] & \mathbf{Mod}_S \arrow[l, "\mathbf{res}\,u"'] \end{tikzcd}$$

organized into a sequence of adjunctions

$$\mathbf{ext}\,u \;\dashv\; \mathbf{res}\,u \;\dashv\; \mathbf{coext}\,u$$

where extension of scalar $\mathbf{ext}_u$ is left adjoint, and coextension of scalar $\mathbf{coext}_u$, right adjoint to restriction of scalar $\mathbf{res}_u$.

Ringed categories

A **ringed category** is as a pair $(\mathscr{C}, \pi)$ consisting of

- ▷ a category $\mathscr{C}$,
- ▷ a functor $\pi : \mathscr{C} \to \mathbf{Ring}$ to the category of commutative rings.

Typically, the category $\mathbf{Mod}$ defines a ringed category, with functor:

$$\pi \quad : \quad \mathbf{Mod} \longrightarrow \mathbf{Ring}$$

The slice 2-category $\mathbf{Cat}/\mathbf{Ring}$ has ringed categories as objects, fibrewise functors and natural transformations as 1-cells and 2-cells.

The 2-category $\mathbf{Cat}/\mathbf{Ring}$ is cartesian, with cartesian product defined by the expected pullback above $\mathbf{Ring}$.

$\mathbf{Mod}$ as a symmetric monoidal ringed category

The cartesian product of $\mathbf{Mod}$ with itself is computed by the pullback:

$$\begin{tikzcd} \mathbf{Mod} \times_{\mathbf{Ring}} \mathbf{Mod} \arrow[r] \arrow[d] & \mathbf{Mod} \arrow[d, "\pi"] \\ \mathbf{Mod} \arrow[r, "\pi"] & \mathbf{Ring} \end{tikzcd}$$

and comes equipped with a fibrewise tensor product

$$\otimes_{\mathbf{Mod}} \quad : \quad \mathbf{Mod} \times_{\mathbf{Ring}} \mathbf{Mod} \longrightarrow \mathbf{Mod}$$

which transports every pair of modules on the same ring R

$$(R, M) \qquad\qquad (R, N)$$

to the R-module $(R, M \otimes_R N)$ defined by their tensor product in $\mathbf{Mod}_R$.

$\mathbf{Mod}$ as a symmetric monoidal ringed category

The functor $\otimes_{\mathbf{Mod}}$ transports every pair of module homomorphisms

$$u : R \longrightarrow S \quad \models \quad h_1 : M_1 \longrightarrow N_1$$
$$u : R \longrightarrow S \quad \models \quad h_2 : M_2 \longrightarrow N_2$$

above the same ring homomorphism $u : R \to S$ to the homomorphism

$$u : R \longrightarrow S \models h_1 \otimes_u h_2 : M_1 \otimes_R M_2 \longrightarrow N_1 \otimes_S N_2$$

where $h_1 \otimes_u h_2$ is the unique map making the diagram commute:

$$\begin{tikzcd}
M_1 \otimes R \otimes M_2 \arrow[r, "h_1 \otimes u \otimes h_2"] \arrow[d, shift right, "\mathrm{act}_{M_1} \otimes M_2"'] \arrow[d, shift left, "M_1 \otimes \mathrm{act}_{M_2}"] & N_1 \otimes S \otimes N_2 \arrow[d, shift right, "\mathrm{act}_{N_1} \otimes N_2"'] \arrow[d, shift left, "N_1 \otimes \mathrm{act}_{N_2}"] \\
M_1 \otimes M_2 \arrow[r, "h_1 \otimes h_2"] \arrow[d, "\textit{quotient map}"'] & N_1 \otimes N_2 \arrow[d, "\textit{quotient map}"] \\
M_1 \otimes_R M_2 \arrow[r, "h_1 \otimes_u h_2"] & N_1 \otimes_S N_2
\end{tikzcd}$$

$\mathbf{Mod}$ as a symmetric monoidal ringed category

In this way, the ring category

$$\pi \quad : \quad \mathbf{Mod} \longrightarrow \mathbf{Ring}$$

defines a **symmetric pseudomonoid** in the 2-category $\mathbf{Cat}/\mathbf{Ring}$.

This is what we call a **symmetric monoidal ringed category**.

Note that the fibrewise unit of $(\mathbf{Mod}, \pi)$ is defined as the functor

$$1_{\mathbf{Mod}} \quad : \quad \mathbf{Ring} \longrightarrow \mathbf{Mod}$$

which transports every commutative ring R into itself, seen as an R-module.

Functors of points and $\mathbf{Ring}$-spaces

A **$\mathbf{Ring}$-space** is defined as a covariant presheaf

$$X \quad : \quad \mathbf{Ring} \longrightarrow \mathbf{Set}$$

on the category $\mathbf{Ring}$ of commutative rings,

To every such $\mathbf{Ring}$-space X, we associate its Grothendieck category

$$\mathbf{Points}(X)$$

- whose objects are the pairs (R, x) with $x \in X(R)$
- whose maps $u : (R, x) \to (S, y)$ are ring homomorphisms $u : R \to S$ transporting the element $x \in X(R)$ to the element $y \in X(S)$, in the sense that

$$X(u)(x) \quad = \quad y.$$

Functors of points and **Ring**-spaces

The category $\mathbf{Points}(X)$ comes equipped with a **functor of point**

$$\pi_X \quad : \quad \mathbf{Points}(X) \longrightarrow \mathbf{Ring}$$

and thus defines a ringed category.

A map $f : X \to Y$ of $\mathbf{Ring}$-spaces may be equivalently defined as a functor

$$f \quad : \quad \mathbf{Points}(X) \longrightarrow \mathbf{Points}(Y)$$

making the diagram commute:

$$\begin{tikzcd} \mathbf{Points}(X) \arrow[rr, "f"] \arrow[dr, bend right, "\pi_X"'] & & \mathbf{Points}(Y) \arrow[dl, bend left, "\pi_Y"] \\ & \mathbf{Ring} & \end{tikzcd}$$

thus defining a functor of ringed categories.

Presheaves of modules

A **presheaf of modules** M on a $\mathbf{Ring}$-space

$$X \quad : \quad \mathbf{Ring} \longrightarrow \mathbf{Set}$$

or more simply, **an $\mathcal{O}_X$-module** M, consists of the following data:

- ▷ for each point $(R, x) \in \mathbf{Points}(X)$, a module $M_x \in \mathbf{Mod}_R$ over the ring R,
- ▷ for each map $u : (R, x) \to (S, y)$ in $\mathbf{Points}(X)$, a module homomorphism

$$u \;:\; R \longrightarrow S \quad \models \quad \theta(u, x) \;:\; M_x \longrightarrow N_y$$

living over the ring homomorphism $u : R \to S$.

Adapted from Demazure-Gabriel (1970) and Kontsevich-Rosenberg (2004).

Presheaves of modules

The map θ is required to satisfy the following functorial properties:

1. first of all, the identity on the point (R, x) in the category $\mathbf{Points}(X)$ is transported to the identity map on the associated R-module:

$$id_R \quad \models \quad \theta(id_{(R,x)}) = id_{M_x}$$

2. then, given two maps

$$(u, x) : (R, x) \longrightarrow (S, y) \qquad\qquad (v, y) : (S, y) \longrightarrow (T, z)$$

in the category $\mathbf{Points}(X)$, one has:

$$v \circ u \quad \models \quad \theta((v, y) \circ (u, x)) = \theta(v, y) \circ \theta(u, x)$$

where composition is computed in the ringed category $\mathbf{Points}(X) \to \mathbf{Ring}$.

Presheaves of modules

In the sequel, we will use the following equivalent formulation:

Proposition. An $\mathscr{O}_X$-module M is the same thing as a functor

$$M \quad : \quad \mathbf{Points}(X) \longrightarrow \mathbf{Mod}$$

making the diagram below commute:

$$\begin{tikzcd}
\mathbf{Points}(X) \arrow[rr, "M"] \arrow[dr, "\pi_X"'] & & \mathbf{Mod} \arrow[dl, "\pi"] \\
& \mathbf{Ring} &
\end{tikzcd}$$

Note that Kontsevich and Rosenberg (2004) use this specific formulation of presheaves of modules in their work on noncommutative geometry.

The structure presheaf of modules

Every $\mathbf{Ring}$-space

$$X \quad : \quad \mathbf{Ring} \longrightarrow \mathbf{Set}$$

comes equipped with a specific presheaf of module, called the **structure presheaf of modules**, and defined as the composite

$$\mathscr{O}_X \quad : \quad \mathbf{Points}(X) \xrightarrow{\ \pi_X\ } \mathbf{Ring} \xrightarrow{\ \mathscr{O}\ } \mathbf{Mod}$$

where the functor

$$\mathscr{O} = 1_{\mathbf{Mod}} \quad : \quad \mathbf{Ring} \to \mathbf{Mod}$$

denotes the section of $\pi : \mathbf{Mod} \to \mathbf{Ring}$ which transports every commutative ring R to itself, seen as an R-module.

The category $\mathbf{PshMod}$ of presheaves of modules and forward morphisms

A **forward morphism** between presheaves of modules

$$(f, \varphi) \quad : \quad (X, M) \longrightarrow (Y, N)$$

is a morphism (= natural transformation) of $\mathbf{Ring}$-spaces $f : X \to Y$ together with a natural transformation

$$\begin{tikzcd}
\mathbf{Points}(X) \arrow[rr, "\mathbf{Points}(f)"] \arrow[dr, "M"'] & \arrow[d, phantom, "\overset{\varphi}{\Longrightarrow}"] & \mathbf{Points}(Y) \arrow[dl, "N"] \\
& \mathbf{Mod} &
\end{tikzcd}$$

The category $\mathbf{PshMod}$ of presheaves of modules and forward morphisms

The natural transformation φ is also required to be vertical (or fibrewise) above $\mathbf{Ring}$, in the sense that the natural transformation

$$
\begin{tikzcd}
\mathbf{Points}(X) \arrow[rr, "\mathbf{Points}(f)"] \arrow[rr, phantom, shift right=4ex, "\overset{\varphi}{\Longrightarrow}"] \arrow[dr, "M"'] \arrow[ddr, bend right, "\pi_X"'] \arrow[ddr, phantom, shift left=2ex, "\Uparrow id"] & & \mathbf{Points}(Y) \arrow[dl, "N"] \arrow[ddl, bend left, "\pi_Y"] \arrow[ddl, phantom, shift right=2ex, "\Downarrow id"] \\
& \mathbf{Mod} \arrow[d, "\pi"] & \\
& \mathbf{Ring} &
\end{tikzcd}
$$

coincides with the identity natural transformation from π_X to $\pi_Y \circ f$.

The category $\mathbf{PshMod}$ of presheaves of modules and forward morphisms

There is an obvious functor

$$\mathbf{p} \quad : \quad \mathbf{PshMod} \longrightarrow [\mathbf{Ring}, \mathbf{Set}]$$

which transports every presheaf of modules (X, M) to its underlying $\mathbf{Ring}$-space X, and every forward morphism $(f, \varphi) : (X, M) \to (Y, N)$ to its underlying morphism $f : X \to Y$ between $\mathbf{Ring}$-spaces.

We thus find convenient to write

$$f \; : \; X \longrightarrow Y \quad \models \quad \varphi \; : \; M \longrightarrow N$$

for a forward morphism between presheaves of modules

$$(f, \varphi) : (X, M) \to (Y, N)$$

The functor $\mathbf{p}$ is a Grothendieck fibration

Every morphism $f : X \to Y$ of $\mathbf{Ring}$-spaces X and Y induces a functor

$$f^* \quad : \quad \mathbf{PshMod}_Y \longrightarrow \mathbf{PshMod}_X$$

which transports every $\mathscr{O}_Y$-module N into the $\mathscr{O}_X$-module $N \circ \mathbf{Points}(f)$ obtained by precomposition with the functor $\mathbf{Points}(f)$, as depicted below:

$$\begin{tikzcd}
\mathbf{Points}(X) \arrow[rr, "\mathbf{Points}(f)"] \arrow[dr, bend right, "\pi_X"'] & & \mathbf{Points}(Y) \arrow[dl, "N"'] \arrow[dl, bend left, "\pi_Y"] \\
& \mathbf{Mod} \arrow[d, "\pi"] & \\
& \mathbf{Ring} &
\end{tikzcd}$$

An axiomatic approach to abelian groups (3)

Here, we make the extra assumption that

the category $\mathbf{Ring}$

as well as

every category $\mathbf{Mod}_R$ associated to a commutative ring R

has **all small colimits**.

The property holds in the case of the category $\mathscr{A} = \mathbf{Ab}$ of abelian groups.

The functor $\mathbf{p}$ is a Grothendieck bifibration

In that case, it turns out that the functor

$$\mathbf{p} \quad : \quad \mathbf{PshMod} \longrightarrow [\mathbf{Ring}, \mathbf{Set}]$$

is also a Grothendieck bifibration, but for less immediate reasons.

For every morphism $f : X \to Y$ between $\mathbf{Ring}$-spaces, the functor

$$f^* \quad : \quad \mathbf{PshMod}_Y \longrightarrow \mathbf{PshMod}_X$$

has a left adjoint

$$\exists_f \quad : \quad \mathbf{PshMod}_X \longrightarrow \mathbf{PshMod}_Y$$

The functor $\mathbf{p}$ is a Grothendieck bifibration

It is worth noting that the $\mathscr{O}_Y$-module $\exists_f(M)$ can be directly described with an explicit formula:

$$\exists_f(M) \;:\; y \in Y(R) \mapsto \bigoplus_{\{x \in X(R), fx = y\}} M_x \in \mathbf{Mod}_R.$$

The adjunction $\exists_f \dashv f^*$ gives rise to a sequence of natural bijections, which can be formulated in the type-theoretic fashion of PAM-Zeilberger (2015)

$$\dfrac{\dfrac{id_X : X \to X \models M \to f^*(N)}{f : X \to Y \models M \to N}}{id_Y : Y \to Y \models \exists_f(M) \to N}$$

The category $\mathbf{PshMod}^{\ominus}$ of presheaf of modules and backward morphisms

A **backward morphism** between presheaves of modules

$$(f, \psi) \quad : \quad (X, M) \longrightarrow (Y, N)$$

is a morphism (= natural transformation) of $\mathbf{Ring}$-spaces $f : X \to Y$ together with a natural transformation

$$\begin{tikzcd}
\mathbf{Points}(X) \arrow[rr, "f"] \arrow[dr, "M"'] & {} \arrow[d, phantom, "\overset{\psi}{\Longleftarrow}"] & \mathbf{Points}(Y) \arrow[dl, "N"] \\
& \mathbf{Mod} &
\end{tikzcd}$$

The category $\mathbf{PshMod}^{\ominus}$ of presheaf of modules and backward morphisms

One requires moreover that ψ is vertical in the sense that the diagram below commutes:

$$\begin{tikzcd}
\mathbf{Points}(X) \arrow[rr, "f"] \arrow[dr, "M"'] \arrow[ddr, bend right, "\pi_X"'] & & \mathbf{Points}(Y) \arrow[dl, "N"] \arrow[ddl, bend left, "\pi_Y"] \arrow[ll, phantom, shift right=3, "\overset{\psi}{\Leftarrow}"] \\
& \mathbf{Mod} \arrow[d, "\pi"] & \\
& \mathbf{Ring} &
\end{tikzcd}$$

The category $\mathbf{PshMod}^{\ominus}$ of presheaf of modules and backward morphisms

The category $\mathbf{PshMod}^{\ominus}$ has presheaves of modules as objects, and backward morphism as morphisms. There is an obvious functor

$$\mathbf{p}^{\ominus} \quad : \quad \mathbf{PshMod}^{\ominus} \longrightarrow [\mathbf{Ring}, \mathbf{Set}]$$

We thus find convenient to write

$$f : X \longrightarrow Y \quad \models^{op} \quad \psi \quad : \quad M \longrightarrow N$$

for such a backward morphism $(f, \psi) : (X, M) \to (Y, N)$ between presheaves of modules.

An axiomatic approach to abelian groups (4)

Here, we make the extra assumption that

the category $\mathbf{Ring}$

as well as

every category $\mathbf{Mod}_R$ associated to a commutative ring R

has **all small limits**.

The property holds in the case of the category $\mathscr{A} = \mathbf{Ab}$ of abelian groups.

The functor $\mathbf{p}^{\ominus}$ is a Grothendieck bifibration

As the opposite of the fibration $\mathbf{p}$, the functor

$$\mathbf{p}^{\ominus} \quad : \quad \mathbf{PshMod}^{\ominus} \longrightarrow [\mathbf{Ring}, \mathbf{Set}]$$

is also a Grothendieck fibration with the opposite functor

$$(f^{*})^{op} \quad : \quad \mathbf{PshMod}_{Y}^{op} \longrightarrow \mathbf{PshMod}_{X}^{op}$$

as pullback functor associated to a morphism $f : X \to Y$ of $\mathbf{Ring}$-spaces.

Fact. There is a **functor**

$$\forall_{f} \quad : \quad \mathbf{PshMod}_{X} \longrightarrow \mathbf{PshMod}_{Y}.$$

right adjoint to the functor f^{*}.

By duality, the functor $(\forall_{f})^{op}$ is left adjoint to the functor $(f^{*})^{op}$.

The functor $\mathbf{p}$ is a Grothendieck trifibration

The adjunction $f^* \dashv \forall_f$ gives rise to a sequence of natural bijections, formulated below in the type-theoretic fashion:

$$\cfrac{\cfrac{id_X : X \to X \models^{op} M \to f^*(N)}{f : X \to Y \models^{op} M \to N}}{id_Y : Y \to Y \models^{op} \forall_f(M) \to N}$$

In summary, every morphism $f : X \to Y$ between $\mathbf{Ring}$-spaces X and Y induces three functors

$$\begin{tikzcd} \mathbf{PshMod}_X \arrow[r, shift left=8, "\forall_f"] \arrow[r, shift right=8, "\exists_f"'] & \mathbf{PshMod}_Y \arrow[l, "f^*" description] \end{tikzcd}$$

organized into a sequence of adjunctions

$$\exists_f \dashv f^* \dashv \forall_f.$$

The category $\mathbf{PshMod}$ is symmetric monoidal closed above the cartesian closed category $[\mathbf{Ring}, \mathbf{Set}]$

The presheaf category $[\mathbf{Ring}, \mathbf{Set}]$ of $\mathbf{Ring}$-spaces is cartesian closed.

We exhibit a **symmetric monoidal closed structure** on $\mathbf{PshMod}$ designed in such a way that the functor

$$\mathbf{p} \quad : \quad \mathbf{PshMod} \longrightarrow [\mathbf{Ring}, \mathbf{Set}]$$

is **symmetric monoidal closed**.

The cartesian structure on $[\mathbf{Ring}, \mathbf{Set}]$

Suppose given a pair of $\mathbf{Ring}$-spaces

$$X, Y \quad : \quad \mathbf{Ring} \longrightarrow \mathbf{Set}$$

and a pair of presheaves of modules M and N over them:

$$M \in \mathbf{PshMod}_X \qquad\qquad N \in \mathbf{PshMod}_Y.$$

The cartesian product $X \times Y$ of $\mathbf{Ring}$-spaces is defined pointwise:

$$X \times Y \quad : \quad R \;\mapsto\; X(R) \times Y(R).$$

The monoidal structure on $\mathbf{PshMod}$

The tensor product

$$M \otimes N \in \mathbf{PshMod}_{X \times Y}$$

is defined using the isomorphism:

$$\mathbf{Points}(X \times Y) \;\cong\; \mathbf{Points}(X) \times_{\mathbf{Ring}} \mathbf{Points}(Y)$$

as the presheaf of modules

$$\mathbf{Points}(X \times Y) \xrightarrow{(M,N)} \mathbf{Mod} \times_{\mathbf{Ring}} \mathbf{Mod} \xrightarrow{\otimes} \mathbf{Mod}$$

where the functor (M, N) is defined by universality of the pullback.

The monoidal structure on $\mathbf{PshMod}$

The unit of the tensor product is the structure presheaf of modules

$$(\operatorname{Spec}\mathbb{Z}, \mathscr{O}_{\operatorname{Spec}\mathbb{Z}}) \quad : \quad (R, *_R) \mapsto R \in \mathbf{Mod}_R$$

on the terminal object $\operatorname{Spec}\mathbb{Z}$ of the category $[\mathbf{Ring}, \mathbf{Set}]$.

Here, $*_R$ denotes the unique element of the singleton set $\operatorname{Spec}\mathbb{Z}(R)$.

The closed structure on $\mathbf{PshMod}$

The internal hom $X \Rightarrow Y$ in $[\mathbf{Ring}, \mathbf{Set}]$ is the covariant presheaf

$$X \Rightarrow Y \quad : \quad \mathbf{Ring} \longrightarrow \mathbf{Set}$$

which associates to every commutative ring R the set

$$X \Rightarrow Y \quad : \quad R \;\mapsto\; ([\mathbf{Ring}, \mathbf{Set}]/\mathbf{y}_R)(\mathbf{y}_R \times X, \mathbf{y}_R \times Y)$$

of natural transformations f making the diagram commute:

$$\begin{tikzcd}
\mathbf{y}_R \times X \arrow[rr, "f"] \arrow[dr, "\pi_{R,X}"'] & & \mathbf{y}_R \times Y \arrow[dl, "\pi_{R,Y}"] \\
& \mathbf{y}_R &
\end{tikzcd}$$

The closed structure on $\mathbf{PshMod}$

Here,

$$\mathbf{y}_R \in [\mathbf{Ring}, \mathbf{Set}]$$

denotes the Yoneda presheaf

$$\mathbf{y}_R \quad : \quad S \mapsto \mathbf{Ring}(R, S) \quad : \quad \mathbf{Ring} \longrightarrow \mathbf{Set}$$

generated by the commutative ring R, while

$$\pi_{R,X} \quad : \quad \mathbf{y}_R \times X \longrightarrow \mathbf{y}_R$$

$$\pi_{R,Y} \quad : \quad \mathbf{y}_R \times Y \longrightarrow \mathbf{y}_S$$

denote the first projections in the cartesian category $[\mathbf{Ring}, \mathbf{Set}]$.

The closed structure on $\mathbf{PshMod}$

The presheaf of modules

$$M \multimap N \in \mathbf{PshMod}_{(X \Rightarrow Y)}$$

is constructed in the following way. To every element

$$f \in (X \Rightarrow Y)(R)$$

we associate the R-module

$$(M \multimap N)_f$$

consisting of all natural transformations φ making the diagram commute:

The closed structure on $\mathbf{PshMod}$

$$
\begin{tikzcd}
\mathbf{Points}(\mathbf{y}_R \times X) \arrow[rr, "f"] \arrow[d, "\mathbf{Points}(\pi_2)"'] \arrow[drr, phantom, "\overset{\varphi}{\Longrightarrow}"] & & \mathbf{Points}(\mathbf{y}_R \times Y) \arrow[d, "\mathbf{Points}(\pi_2)"] \\
\mathbf{Points}(X) \arrow[dr, "M"'] \arrow[ddr, bend right, "\pi_X"'] & & \mathbf{Points}(Y) \arrow[dl, "N"] \arrow[ddl, bend left, "\pi_Y"] \\
& \mathbf{Mod} \arrow[d, "\pi"] & \\
& \mathbf{Ring} &
\end{tikzcd}
$$

The closed structure on $\mathbf{PshMod}$

The R-module

$$(M \multimap N)_f \quad \in \quad \mathbf{Mod}_R$$

associated to the map of $\mathbf{Ring}$-space

$$f \quad : \quad \mathbf{y}_R \times X \longrightarrow \mathbf{y}_R \times Y$$

can be computed using the **end formula**

$$(M \multimap N)_f \quad = \quad \int_{(u:R\to S, x\in X(S))\in \mathbf{Points}(\mathbf{y}_R \times X)} \mathbf{res}_u\,(\,[M_x, N_{f(u,x)}]_S\,)$$

in the category $\mathbf{Mod}_R$.

Main result of the talk

Theorem. The tensor product

$$M, N \mapsto M \otimes N$$

and the implication just defined

$$M, N \mapsto M \multimap N$$

equip $\mathbf{PshMod}$ with the structure of a **symmetric monoidal category**.

This structure is moreover transported by the functor

$$\mathbf{p} \quad : \quad \mathbf{PshMod} \longrightarrow [\mathbf{Ring}, \mathbf{Set}]$$

to the cartesian closed structure of $[\mathbf{Ring}, \mathbf{Set}]$ in the sense that

$$\mathbf{p}(M \otimes N) = X \times Y \qquad\qquad \mathbf{p}(M \multimap N) = X \Rightarrow Y$$

for the $\mathbf{Ring}$-spaces $X = \mathbf{p}(M)$ and $Y = \mathbf{p}(N)$.

Application: $\mathbf{PshMod}_X$ is a smcc

We establish that the category $\mathbf{PshMod}_X$ associated to a $\mathbf{Ring}$-space

$$X \quad : \quad \mathbf{Ring} \longrightarrow \mathbf{Set}$$

is **symmetric monoidal closed**. The tensor product $M \otimes_X N$ of a pair of $\mathscr{O}_X$-modules M, N is defined as

$$M \otimes_X N \quad := \quad \Delta^*(M \otimes N)$$

where we use the notation

$$\Delta \quad : \quad X \longrightarrow X \times X$$

for the diagonal map induced by the cartesian structure of the presheaf category $[\mathbf{Ring}, \mathbf{Set}]$. The tensorial unit is defined as the structure presheaf of modules $\mathscr{O}_X$ associated to the $\mathbf{Ring}$-space X.

Application: $\mathbf{PshMod}_X$ is a smcc

The internal hom $M \multimap_X N$ of a pair of $\mathscr{O}_X$-modules M, N is defined as

$$M \multimap_X N \quad := \quad curry^*(M \multimap \forall_\Delta(N))$$

where

$$curry \quad : \quad X \longrightarrow X \Rightarrow (X \times X)$$

is the map obtained by currifying the identity map

$$id_{X \times X} \quad : \quad X \times X \longrightarrow X \times X$$

on the second component X. One obtains that

Proposition. The category $\mathbf{PshMod}_X$ equipped with $\otimes_X$ and $\multimap_X$ defines a symmetric monoidal closed category.

Proof in a nutshell

$$
\begin{array}{c}
id_X : X \to X \models (M \otimes_X N) \to P \\ \hline
id_X : X \to X \models \Delta^*(M \otimes N) \to P \\ \hline
id_X : X \to X \models^{op} P \to \Delta^*(M \otimes N) \\ \hline
\Delta : X \to X \times X \models^{op} P \to M \otimes N \\ \hline
id_{X \times X} : X \times X \to X \times X \models^{op} \forall_\Delta(P) \to M \otimes N \\ \hline
id_{X \times X} : X \times X \to X \times X \models M \otimes N \to \forall_\Delta(P) \\ \hline
curry : X \to X \Rightarrow (X \times X) \models N \to M \multimap \forall_\Delta(P) \\ \hline
id_X : X \to X \models N \to curry^*(M \multimap \forall_\Delta(P)) \\ \hline
id_X : X \to X \models N \to (M \multimap_X P)
\end{array}
$$

Sequence of natural bijections establishing that the functor

$$M \otimes_X - \quad : \quad \mathbf{PshMod}_X \longrightarrow \mathbf{PshMod}_X$$

is left adjoint to the functor

$$M \multimap_X - \quad : \quad \mathbf{PshMod}_X \longrightarrow \mathbf{PshMod}_X$$

for any presheaf of modules $M \in \mathbf{PshMod}_X$.

Application: change-of-basis functors

Moreover, given a morphism $X \to Y$ in $[\mathbf{Ring}, \mathbf{Set}]$ and two $\mathscr{O}_Y$-modules M and N, the fact that $\Delta_Y \circ f = (f \times f) \circ \Delta_X$ and the isomorphism

$$(f \times f)^*(M \otimes N) \cong f^*(M) \otimes f^*(N)$$

imply that

$$f^* \ : \ \mathbf{PshMod}_Y \longrightarrow \mathbf{PshMod}_X$$

defines a **strongly monoidal functor**, in the sense that there exists a family of isomorphisms

$$m_{X,M,Y,N} \ : \ f^*(M) \otimes_X f^*(N) \xrightarrow{\ \sim\ } f^*(M \otimes_Y N)$$

$$m_{X,Y} \ : \ \mathscr{O}_X \xrightarrow{\ \sim\ } f^*(\mathscr{O}_Y)$$

making the expected coherence diagrams commute.

Application: change-of-basis functors

From this follows that

- ▷ the right adjoint functor $\forall_f$ is lax symmetric monoidal ;
- ▷ the adjunction $f^* \dashv \forall_f$ is lax symmetric monoidal ;
- ▷ the left adjoint functor $\exists_f$ is oplax symmetric monoidal ;
- ▷ the adjunction $\exists_f \dashv f^*$ is oplax symmetric monoidal.

In particular, the two functors $\forall_f$ and $\exists_f$ come with families of maps:

$$\forall_f(M) \otimes_N \forall_f(Y) \longrightarrow \forall_f(M \otimes_X N) \qquad \mathscr{O}_Y \longrightarrow \forall_f(\mathscr{O}_X)$$

$$\exists_f(M \otimes_X N) \longrightarrow \exists_f(M) \otimes_Y \exists_f(N) \qquad \exists_f(\mathscr{O}_X) \longrightarrow \mathscr{O}_Y$$

parametrized by $\mathscr{O}_X$-modules M and N.

What we did not speak about here

- the **Sweedler dual construction** of a free commutative coalgebra

$$\begin{tikzcd} \mathbf{Mod} \arrow[r, bend right, "\mathbf{Sym}"'] \arrow[r, phantom, "\top"] & \mathbf{Alg} \arrow[l, bend right, "\mathbf{Forget}"'] \end{tikzcd}$$

- the induced construction of an **linear-non-linear** adjunction

$$\begin{tikzcd} \mathbf{PshCoAlg}_X \arrow[r, bend right, "\mathbf{Lin}_X"'] \arrow[r, phantom, "\top"] & \mathbf{PshMod}_X \arrow[l, bend right, "\mathbf{Exp}_X"'] \end{tikzcd}$$

defining an **exponential modality** $A \mapsto !A$ for linear logic.

Conclusion and future directions

- work with **sheaves** and **schemes** instead of general presheaves,
- understand the structure of the inclusion functor

$$\mathbf{qcMod}_X \longrightarrow \mathbf{PshMod}_X$$

  from the category $\mathbf{qcMod}_X$ of **quasi-coherent modules**.
- shift to **derived categories** and clarify the connection

  linear logic $\leftrightarrow$ **Grothendieck-Verdier duality**
- explore the connection to **dependent and homotopy type theory**.

Thank you !

The closed structure on $\mathbf{PshMod}$

This condition may be expanded using the notation $f(u, x) = (u, \tilde{f}(u, x))$.

Such a natural transformation φ is a family of module homomorphisms

$$id_S \quad : \quad S \longrightarrow S \quad \models \quad \varphi_{u,x} \quad : \quad M_x \longrightarrow N_{\tilde{f}(u,x)}$$

for $u : R \to S$ and $x \in X(S)$, natural in u and x in the sense that the diagram

$$\begin{tikzcd} M_x \arrow[r, "\varphi_{u,x}"] \arrow[d, "M_v"'] & N_{\tilde{f}(u,x)} \arrow[d, "N_{\tilde{f}(v,v)}"] \\ M_{x'} \arrow[r, "\varphi_{v\circ u,x'}"] & N_{\tilde{f}(v\circ u,x')} \end{tikzcd}$$

commutes for every ring homomorphism $v : S \to S'$ with $X(v)(x) = x'$.