A Coherator For Semi-Cubical Weak ω-Categories

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Introduction (I)

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  - existence of cells up to a certain level
  - strict vs. weak
  - various basic shapes: globes, simplices, cubes, opetopes, ...

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- Globular weak ω-categories
  - Batanin-Leinster: Algebras for the initial globular operad with contraction.
  - Maltsiniotis (after Grothendieck): Defined by a *coherator*.
  - Ara (a bit of help from Bourke): The Grothendieck-Maltsiniotis definition can specialize to the Batanin-Leinster one.

Introduction (II)

- Aim: Define a coherator for semi-cubical weak ω-categories à la Grothendieck-Maltsiniotis (WIP).
  - weak ω-categories based on the category of semi-cubes
  - Unpublished work I did during my PhD (circa. 2019/2020)
  - Please give me your feedback

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- Cubical weak ω-categories
  - Kachour: weak ω-categories on reflexive cubes à la Batanin-Leinster, but newer publications getting closer to the Grothendieck-Maltsiniotis style.
  - Grandis: cubical categories with symmetries.

Grothendieck-Maltsiniotis globular weak ω-categories

Globes and Globular Sets

- The *category of globes* $\mathbb{G}$:

$$0 \overset{\sigma}{\underset{\tau}{\rightrightarrows}} 1 \overset{\sigma}{\underset{\tau}{\rightrightarrows}} 2 \overset{\sigma}{\underset{\tau}{\rightrightarrows}} \dots \qquad \sigma\sigma = \tau\sigma \quad \sigma\tau = \tau\tau$$

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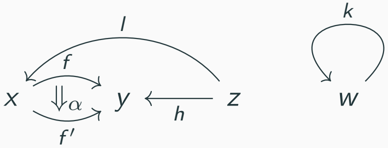

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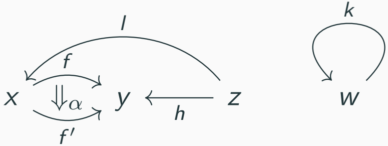

- *Disks* are the representable presheaves

Pasting Schemes/Globular Sums

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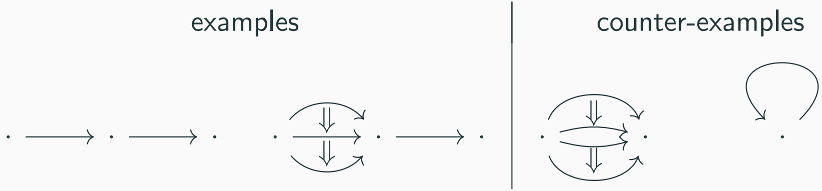

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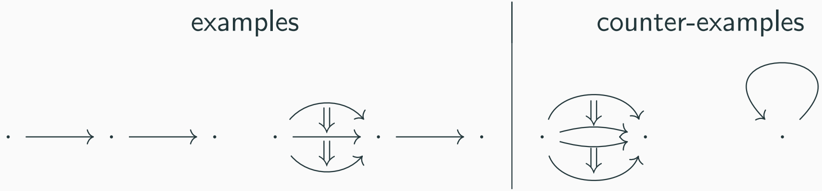
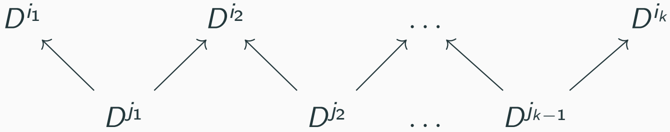


- Formally, they are obtained as *globular sums*, i.e., limits of the following form



Source and Target of a Pasting Scheme

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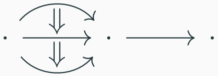
Example

P



∂P

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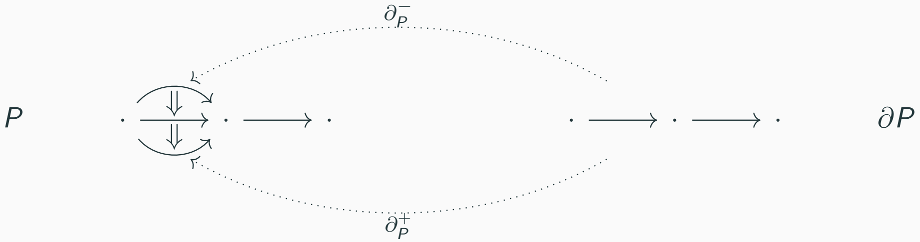

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- *Globular extension*: a category $\mathcal{C}$ equipped with a functor $\mathbb{G} \to \mathcal{C}$ such that $\mathcal{C}$ has the globular sums.

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  Intuition: a globular theory C contain pasting schemes with operations producing extra cells.
- Reminder on Yoneda Lemma: cells in $P \leftrightsquigarrow$ maps $D^n \to P$ in C

Coherator for Globular Weak ω-categories

- Given an object P in a globular theory $\mathcal{C}$, a cell x is *algebraic*, if there are no non-trivial map $f : Q \to P$ in Θ_0 such that x is in the image of f.
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$$\Theta_\infty = \lim(\Theta_0 \to \Theta_1 \to \Theta_2 \to \ldots)$$

  where Θ_{n+1} is formally obtained from Θ_n by universally adding a lift for every pair of cells (x, y) in P which either:
  - write as $(\partial_X^-(x'), \partial_X^+(y'))$ with x', y' algebraic in ∂P
  - are both algebraic in P

  and for which a lift was not added at an earlier stage.

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    There exists a cell witnessing the composition of X from x to y

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    - Adding a lift for every pair (x, y) that are algebraic
      Any two compositions of X are related by a higher cell: weak uniqueness
- Existence + weak uniqueness related with contractibility in Topology/HoTT.

Identities, Compositions, Associators

We consider a weak ω-category X:

- A 0-cell x defines a map $x : D^0 \to X$. D^0 has a unique cell x', which is algebraic, hence the pair x', x' has a lift $\mathrm{id}(x')$, which induces a 1-cell $\mathrm{id}(x) : x \to x$ in X.

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- $D^1 \sqcup_{D^0} D^1 \sqcup_{D^0} D^1 \sqcup_{D^0} D^1$ is given by $x' \xrightarrow{f'} y' \xrightarrow{g'} z' \xrightarrow{h'} w'$, by the previous point, $f' \star_0 (g' \star_0 h')$ and $(f' \star_0 g') \star_0 h'$ both exist, are parallel and are algebraic, hence there exists a cell $\alpha_{f',g',h'} : f' \star_0 (g' \star_0 h') \to (f' \star_0 g') \star_0 h'$. For $x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} w$ in X, this gives $\alpha_{f,g,h} : f \star_0 (g \star_0 h) \to (f \star_0 g) \star_0 h$

Coherator for semi-cubical weak ω-categories

Semi-Cubical Sets and Semi-Cubical Pasting Schemes

- The *category of semi-cubes* $\square$:

$$0 \underset{\tau_0}{\overset{\sigma_0}{\rightrightarrows}} 1 \; \begin{smallmatrix} \xrightarrow{\sigma_1} \\ \xrightarrow{\sigma_0} \\ \xrightarrow{\tau_0} \\ \xrightarrow{\tau_1} \end{smallmatrix} \; 2 \; \begin{smallmatrix} \xrightarrow{\sigma_2} \\ \xrightarrow{\sigma_1} \\ \xrightarrow{\sigma_0} \\ \xrightarrow{\tau_0} \\ \xrightarrow{\tau_1} \\ \xrightarrow{\tau_2} \end{smallmatrix} \; \ldots \qquad \forall j < i, \begin{cases} \sigma_j\sigma_i = \sigma_{i+1}\sigma_j & \sigma_j\tau_i = \tau_{i+1}\sigma_j \\ \tau_j\sigma_i = \sigma_{i+1}\tau_j & \tau_j\tau_i = \tau_{i+1}\tau_i \end{cases}$$

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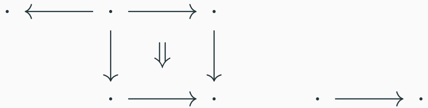

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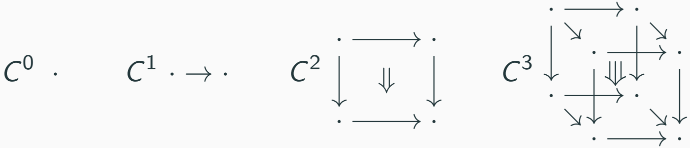

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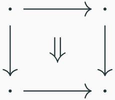
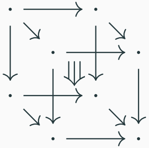
$C^0 \quad \cdot$ $\qquad C^1 \quad \cdot \to \cdot$ $\qquad C^2$

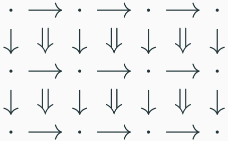


C^3



- Semi-cubical pasting schemes are rectangular grids

  Example:



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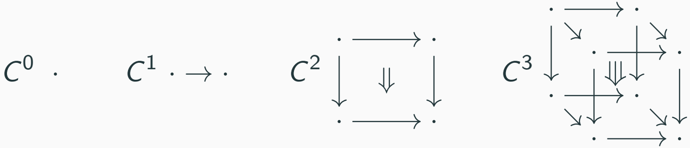

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$$\begin{array}{ccccccc}
\cdot & \longrightarrow & \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot \\
\downarrow & \Downarrow & \downarrow & \Downarrow & \downarrow & \Downarrow & \downarrow \\
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\downarrow & \Downarrow & \downarrow & \Downarrow & \downarrow & \Downarrow & \downarrow \\
\cdot & \longrightarrow & \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot
\end{array}$$

- A semi-cubical pasting scheme P of dimension n has n boundary $\partial_i P$, with maps $\partial^-_{i,P}, \partial^+_{i,P} : \partial_i P \to P$, for $0 \leq i < n$

Semi-Cubical Theories

- Define the *cubical sums*: diagrams for which pasting schemes are limits, and *cubical extension*: a category $\mathcal{C}$ equipped with a functor $\square \to \mathcal{C}$ such that $\mathcal{C}$ has the cubical sums, with the initial cubical extension $\Theta_0^\square$

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- A family of cells $x_1, \ldots, x_n$ in an object P of a cubical theory are simultaneously algebraic if there are no non-trivial map $f : Q \to P$ in $\Theta_0^\square$ such that all the x_i are in the image of f.

Coherator for Semi-Cubical Weak ω-categories

- A family of cells $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ of dimension $n-1$ is *compatible* if the cells fit in the boundary of an n-cube. A lift of a family of compatible cells $x_1, \ldots, x_n, y_1, \ldots, y_n$ of dimension $n-1$ is a cell z of dimension n is a cell z such that $s_i(z) = x_i$ and $t_i(z) = y_i$.

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- The *coherator* $\Theta^{\square}_{\infty}$ is the cubical theory constructed as follows

$$\Theta^{\square}_{\infty} = \lim(\Theta^{\square}_0 \to \Theta^{\square}_1 \to \Theta^{\square}_2 \to \dots)$$

  where $\Theta^{\square}_{n+1}$ is formally obtained from $\Theta^{\square}_n$ by universally adding a lift for every compatible family of cells $(x_1, \dots, x_n, y_1, \dots, y_n)$ in P, where either:
  - we can decompose $x_i = \partial^-_{i,P}(x'_i)$ and $y_i = \partial^+_{i,P}(y'_i)$ with x'_i, y'_i algebraic in $\partial_i P$
  - both families (x_i) and (y_i) are algebraic in P

  and for which a lift was not added at an earlier stage.

Semi-Cubical Weak ω-categories and Operations

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Consider a cubical weak ω-category X:

- For every 0-cell x, we can construct a 1-cell $\mathrm{id}(x) : x \to x$
- For every diagram $x \xrightarrow{f} y \xrightarrow{g} z$, we can construct a 1-cell $f \star_0 g : x \to z$
- For every diagram $x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} w$, we can construct a 2-cell $\begin{array}{ccc} x & \xrightarrow{f\star_0 g} & z \\ {\scriptstyle f}\downarrow & \Downarrow \alpha_{f,g,h} & \downarrow{\scriptstyle h} \\ y & \xrightarrow[g\star_0 h]{} & w \end{array}$

Interchange

- For every diagram $\begin{array}{ccc} x & \xrightarrow{f} & y \\ h\big\downarrow & \Downarrow_{\alpha} & \big\downarrow k \\ x' & \xrightarrow{f'} & y' \\ h'\big\downarrow & \Downarrow_{\alpha'} & \big\downarrow k' \\ x'' & \xrightarrow[f'']{} & y'' \end{array}$, we have a 2-cell $\begin{array}{ccc} x & \xrightarrow{f} & y \\ h \star_0 h'\big\downarrow & \Downarrow_{\alpha \star_1 \alpha'} & \big\downarrow k \star_0 k' \\ x'' & \xrightarrow[f'']{} & y'' \end{array}$

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- For every diagram $\begin{array}{ccccc} x & \xrightarrow{f} & y & \xrightarrow{g} & z \\ h\big\downarrow & \Downarrow_{\alpha} & k & \Downarrow_{\beta} & \big\downarrow l \\ x' & \xrightarrow[f']{} & y' & \xrightarrow[g']{} & z' \end{array}$, we have a 2-cell $\begin{array}{ccc} x & \xrightarrow{f \star_0 g} & z \\ h\big\downarrow & \Downarrow_{\alpha \star_0 \beta} & \big\downarrow l \\ x' & \xrightarrow[f' \star_0 g']{} & z' \end{array}$

Interchange

- For every diagram $\begin{array}{ccc} x & \xrightarrow{f} & y \\ h\downarrow & \Downarrow_{\alpha} & \downarrow k \\ x' & \xrightarrow{f'} & y' \\ h'\downarrow & \Downarrow_{\alpha'} & \downarrow k' \\ x'' & \xrightarrow[f'']{} & y'' \end{array}$, we have a 2-cell $\begin{array}{ccc} x & \xrightarrow{f} & y \\ h \star_0 h' \downarrow & \Downarrow_{\alpha \star_1 \alpha'} & \downarrow k \star_0 k' \\ x'' & \xrightarrow[f'']{} & y'' \end{array}$
- For every diagram $\begin{array}{ccccc} x & \xrightarrow{f} & y & \xrightarrow{g} & z \\ h\downarrow & \Downarrow_{\alpha} & k & \Downarrow_{\beta} & \downarrow l \\ x' & \xrightarrow[f']{} & y' & \xrightarrow[g']{} & z' \end{array}$, we have a 2-cell $\begin{array}{ccc} x & \xrightarrow{f \star_0 g} & z \\ h\downarrow & \Downarrow_{\alpha \star_0 \beta} & \downarrow l \\ x' & \xrightarrow[f' \star_0 g']{} & z' \end{array}$
- $\begin{array}{ccccc} x & \xrightarrow{f} & y & \xrightarrow{g} & z \\ h\downarrow & \Downarrow_{\alpha} & k & \Downarrow_{\beta} & \downarrow l \\ x' & \xrightarrow{f'} & y' & \xrightarrow{g'} & z' \\ h'\downarrow & \Downarrow_{\alpha'} & k' & \Downarrow_{\beta'} & \downarrow l' \\ x'' & \xrightarrow[f'']{} & y'' & \xrightarrow[g'']{} & z'' \end{array}$ gives $(\alpha \star_1 \alpha') \star_0 (\beta \star_1 \beta') \Rrightarrow (\alpha \star_0 \beta) \star_1 (\alpha' \star_0 \beta')$

Interesting Note on Weak Degeneracies

For every 1-cell $x \xrightarrow{f} y$, one can construct two identity 2-cells

- $$\begin{array}{ccc} x & \xrightarrow{f} & y \\ {\scriptstyle \mathrm{id}(x)}\downarrow & \Downarrow \mathrm{id}_1(f) & \downarrow{\scriptstyle \mathrm{id}(y)} \\ x & \xrightarrow[f]{} & y \end{array}$$
- $$\begin{array}{ccc} x & \xrightarrow{\mathrm{id}(x)} & x \\ {\scriptstyle f}\downarrow & \Downarrow \mathrm{id}_0(f) & \downarrow{\scriptstyle f} \\ y & \xrightarrow[\mathrm{id}(y)]{} & y \end{array}$$

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- $$\begin{array}{ccc} x & \xrightarrow{\mathrm{id}(x)} & x \\ {\scriptstyle f}\big\downarrow & \Downarrow \mathrm{id}_0(f) & \big\downarrow{\scriptstyle f} \\ y & \xrightarrow[\mathrm{id}(y)]{} & y \end{array}$$

$\mathrm{id}_1(\mathrm{id}(x))$ and $\mathrm{id}_0(\mathrm{id}(x))$ have the same type, and are equivalent, but not strictly equal!

Thank you!