A Coherator For Semi-Cubical Weak $\omega$-Categories

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Higher categories may come in different flavours

- existence of cells up to a certain level
- strict vs. weak
- various basic shapes: globes, simplices, cubes, opetopes, ...
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\[ \text{today} \]
\[ \omega \]
\[ \text{weak} \]
\[ \text{globes/cubes} \]
Higher categories may come in different flavours

- existence of cells up to a certain level
- strict vs. weak
- various basic shapes: globes, simplices, cubes, opetopes, ...

Globular weak $\omega$-categories

- Batanin-Leinster: Algebras for the initial globular operad with contraction.
- Maltsiniotis (after Grothendieck): Defined by a coherator.
- Ara (a bit of help from Bourke): The Grothendieck-Maltsiniotis definition can specialize to the Batanin-Leinster one.
Aim: Define a coherator for semi-cubical weak \( \omega \)-categories à la Grothendieck-Maltsiniotis (WIP).

- weak \( \omega \)-categories based on the category of semi-cubes
- Unpublished work I did during my PhD (circa. 2019/2020)
- Please give me your feedback
Aim: Define a coherator for semi-cubical weak $\omega$-categories à la Grothendieck-Maltsiniotis (WIP).

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Cubical weak $\omega$-categories

- Kachour: weak $\omega$-categories on reflexive cubes à la Batanin-Leinster, but newer publications getting closer to the Grothendieck-Maltsiniotis style.
- Grandis: cubical categories with symmetries.
Grothendieck-Maltsiniotis globular weak $\omega$-categories
Globes and Globular Sets

- The category of globes $\mathbb{G}$:

$$
\begin{align*}
0 \xrightarrow{\sigma} 1 \xrightarrow{\sigma} 2 \xrightarrow{\sigma} \ldots \\
\sigma \sigma = \tau \sigma & \quad \sigma \tau = \tau \tau
\end{align*}
$$
Globes and Globular Sets

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\[
\begin{array}{cccc}
0 & \xrightarrow{\sigma} & 1 & \xrightarrow{\sigma} & 2 & \xrightarrow{\sigma} & \ldots \\
& & & & & & \\
\tau & & & & & & \\
\end{array}
\]

$\sigma \sigma = \tau \sigma \quad \sigma \tau = \tau \tau$

Globular sets are presheaves on $\mathcal{G}$:

\[
\begin{array}{cccc}
X_0 & \xleftarrow{s} & X_1 & \xleftarrow{s} & X_2 & \xleftarrow{s} & \ldots \\
& & & & & & \\
t & & & & & & \\
\end{array}
\]

$ss = st \quad ts = tt$
Globes and Globular Sets

- **The category of globes** \( \mathbb{G} \):

  \[
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  \end{array}
  \]

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- **Globular sets** are presheaves on \( \mathbb{G} \):

  \[
  X_0 \xleftarrow{s} X_1 \xleftarrow{s} X_2 \xleftarrow{s} \ldots 
  \]

  \( ss = st \quad ts = tt \)

\[\xymatrix{\alpha \ar@/^{0.8pc}/[rr]_{f} & \ar@/^/[r]_{f'} & \ar@{_{(}->}[r]_{l} & y \ar@/^{0.8pc}/[rr]^{kh} & \ar@/^/[r]_{k} & z \ar@/^{0.8pc}/[rr]^{w} & }\]
Globes and Globular Sets

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  \[
  0 \xrightarrow{\sigma} 1 \xrightarrow{\sigma} 2 \xrightarrow{\sigma} \ldots \quad \sigma \sigma = \tau \sigma \quad \sigma \tau = \tau \tau
  \]

- Globular sets are presheaves on $G$:

  
  \[
  X_0 \xleftarrow{s} X_1 \xleftarrow{s} X_2 \xleftarrow{s} \ldots \quad ss = st \quad ts = tt
  \]

- Disks are the representable presheaves

  
  \[
  D^0 : \cdot \quad D^1 : \cdot \rightarrow \cdot \quad D^2 : \cdot \circlearrowleft \cdot \quad D^3 : \cdot \circlearrowright \downarrow \cdot \quad \ldots
  \]
Pasting Schemes/Globular Sums

- **idea**: *Pasting schemes* are the globular sets that should describe a unique composition.
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Formally, they are obtained as globular sums, i.e., limits of the following form:

\[
\begin{align*}
D_{i_1} & \rightarrow D_{i_2} & \cdots & \rightarrow D_{i_k} \\
D_{j_1} & \rightarrow D_{j_2} & \cdots & \rightarrow D_{j_k}
\end{align*}
\]
Pasting Schemes/Globular Sums

**idea**: *Pasting schemes* are the globular sets that should describe a unique composition.

Formally, they are obtained as *globular sums*, i.e., limits of the following form:

\[ D^{i_1} \leftarrow D^{i_2} \leftarrow \ldots \leftarrow D^{i_k} \]

\[ D^{j_1} \rightarrow D^{j_2} \rightarrow \ldots \rightarrow D^{j_{k-1}} \]
Every pasting scheme $P$ has a \textit{boundary} $\partial P$: Formally replace every occurrence of $\text{dim } P$ in the globular sum with $\text{dim } P - 1$
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There exists two maps, called source and target:

$$\partial_P^-, \partial_P^+: \partial P \to P$$
Source and Target of a Pasting Scheme

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Example
Every pasting scheme $P$ has a boundary $\partial P$: Formally replace every occurrence of $\dim P$ in the globular sum with $\dim P - 1$.

There exists two maps, called source and target:

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Example

$$\begin{array}{ccc}
P & \xrightarrow{\partial_P^-} & \partial P \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\partial P & \xleftarrow{\partial_P^+} & P
\end{array}$$
Globular Theories

- **Globular extension**: a category $\mathcal{C}$ equipped with a functor $\mathcal{G} \to \mathcal{C}$ such that $\mathcal{C}$ has the globular sums.

- **The initial globular extension** $\Theta^0$: Explicitly, $\Theta^0$ is the full subcategory of $\mathcal{B}\mathcal{G}$ whose objects are the pasting schemes.

- **Globular theory**: a globular extension $\mathcal{C}$ such that the unique map $\Theta^0 \to \mathcal{C}$ is faithful and identity on objects.

Intuition: a globular theory $\mathcal{C}$ contains pasting schemes with operations producing extra cells.

- **Reminder on Yoneda Lemma**: cells in $\mathcal{P} \leftrightarrow$ maps $D_n \to \mathcal{P}$ in $\mathcal{C}$. 
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- Reminder on Yoneda Lemma: cells in $P \leftrightarrow$ maps $D^n \to P$ in $\mathcal{C}$
Given an object $P$ in a globular theory $C$, a cell $x$ is algebraic, if there are no non-trivial map $f : Q \rightarrow P$ in $\Theta_0$ such that $x$ is in the image of $f$.

Intuition: All maps in $\Theta_0$ are monos $\rightarrow$ algebraic $=$ “uses up” all the data in $P$. 
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A lift of a pair of parallel cells $x,y$ of dimension $n$ is a cell of dimension $n+1$ is a cell $z$ such that $s(z) = x$ and $t(z) = y$. 
Coherator for Globular Weak $\omega$-categories

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The coherator $\Theta_\infty$ is the globular theory constructed as follows

$$\Theta_\infty = \lim(\Theta_0 \to \Theta_1 \to \Theta_2 \to \ldots)$$

where $\Theta_{n+1}$ is formally obtained from $\Theta_n$ by universally adding a lift for every pair of cells $(x, y)$ in $P$ which either:

- write as $(\partial^-_X(x'), \partial^+_X(y'))$ with $x', y'$ algebraic in $\partial P$
- are both algebraic in $P$

and for which a lift was not added at an earlier stage.
Weak $\omega$-categories are presheaves over $\Theta_\infty$ that preserve the globular sums.

- Adding a lift for every pair $(x,y)$ that factor as $\partial^- X(x')$, $\partial^+ X(y')$, with $x'$, $y'$ algebraic.
- There exists a cell witnessing the composition of $X$ from $x$ to $y$.

Any two compositions of $X$ are related by a higher cell: weak uniqueness.

Existence + weak uniqueness related with contractibility in Topology/HoTT.
**Weak $\omega$-categories**

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- Interpretation: Recall that pasting schemes should represent an (essentially) unique way of composing.
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- Existence + weak uniqueness related with contractibility in Topology/HoTT.
We consider a weak $\omega$-category $X$:

- A 0-cell $x$ defines a map $x : D^0 \to X$. $D^0$ has a unique cell $x'$, which is algebraic, hence the pair $x', x'$ has a lift $\text{id}(x')$, which induces a 1-cell $\text{id}(x) : x \to x$ in $X$. 

A diagram $x f \to y g \to z$ in $X$ is an element of $X(D_1 \sqcup D_0 D_1 \sqcup D_0 D_1 \sqcup D_0 D_1)$ (preservation of globular sums).

$D_1 \sqcup D_0 D_1 \sqcup D_0 D_1 \sqcup D_0 D_1$ is given by $x' f' \to y' g' \to z' h'$, and $x'$ is algebraic in the source, $z'$ is algebraic in the target, so there exists a cell $f' \star_0 g' : x' \to z'$, whose image in $X$ is $f \star_0 g : x \to z$.
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Identities, Compositions, Associators
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- $D^1 \sqcup_{D^0} D^1 \sqcup_{D^0} D^1 \sqcup_{D^0} D^1$ is given by $x' \xrightarrow{f'} y' \xrightarrow{g'} z' \xrightarrow{h'} w'$, by the previous point, $f' \star_0 (g' \star_0 h')$ and $(f' \star_0 g') \star_0 h'$ both exist, are parallel and are algebraic, hence there exists a cell $\alpha_{f', g', h'} : f' \star_0 (g' \star_0 h') \to (f' \star_0 g') \star_0 h'$. For $x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} w$ in $X$, this gives $\alpha_{f, g, h} : f \star_0 (g \star_0 h) \to (f \star_0 g) \star_0 h$. 


Coherator for semi-cubical weak $\omega$-categories
The category of semi-cubes □:

\[
\begin{align*}
0 & \rightarrow 1 & 1 & \rightarrow 2 & \cdots \\
\sigma_0 & \rightarrow -\sigma_1 & \rightarrow -\sigma_2 & \rightarrow \\
\tau_0 & \rightarrow -\tau_0 & \rightarrow -\tau_1 & \rightarrow -\tau_2 \\
\end{align*}
\]

\[\forall j < i, \begin{cases} 
\sigma_j \sigma_i = \sigma_{i+1} \sigma_j \\
\sigma_j \tau_i = \tau_{i+1} \sigma_j \\
\tau_j \sigma_i = \sigma_{i+1} \tau_j \\
\tau_j \tau_i = \tau_{i+1} \tau_i \end{cases}\]
The category of semi-cubes □:

\[
\begin{align*}
0 \xrightarrow{\sigma_0} 1 & \xrightarrow{-\sigma_0} 2 \hdots \\
& \xrightarrow{-\sigma_1} 3 \\
& \xrightarrow{-\sigma_2} \hdots \\
\end{align*}
\]

\[\forall j < i, \left\{ \begin{array}{ll}
\sigma_j \sigma_i = \sigma_{i+1} \sigma_j & \sigma_j \tau_i = \tau_{i+1} \sigma_j \\
\tau_j \sigma_i = \sigma_{i+1} \tau_j & \tau_j \tau_i = \tau_{i+1} \tau_i \\
\end{array} \right. \]

Semi-cubical sets are presheaves on □:

\[
\begin{align*}
X_0 \xleftarrow{s_0} \xrightarrow{t_0} X_1 & \xleftarrow{s_0} \xrightarrow{t_0} \hdots \\
& \xleftarrow{s_1} \xrightarrow{t_1} \\
& \xleftarrow{s_2} \xrightarrow{t_2} \hdots \\
\end{align*}
\]

\[\forall j < i, \left\{ \begin{array}{ll}
s_i s_j = s_j s_{i+1} & t_i s_j = s_j t_{i+1} \\
s_i t_j = t_j s_{i+1} & t_i t_j = t_j t_{i+1} \\
\end{array} \right. \]
The category of semi-cubes □:

\[
\begin{align*}
0 & \xrightarrow{\sigma_0} 1 & -\sigma_1 \rightarrow & -\sigma_2 \rightarrow \\
& \xrightarrow{\tau_0} 2 & -\tau_1 \rightarrow & -\tau_2 \rightarrow \\
& \quad \cdots \\
& \forall j < i, \\
& \begin{cases}
\sigma_j \sigma_i = \sigma_{i+1} \sigma_j \\
\sigma_j \tau_i = \tau_{i+1} \sigma_j \\
\tau_j \sigma_i = \sigma_{i+1} \tau_j \\
\tau_j \tau_i = \tau_{i+1} \tau_i
\end{cases}
\end{align*}
\]

Semi-cubical sets are presheaves on □:

\[
\begin{align*}
X_0 & \xleftarrow{s_0} X_1 & s_1 \leftarrow & s_0 \leftarrow \\
& \xrightarrow{t_0} & t_0 \leftarrow & t_1 \leftarrow \\
& \quad \cdots \\
& \forall j < i, \\
& \begin{cases}
s_i s_j = s_j s_{i+1} \\
t_i s_j = s_j t_{i+1} \\
s_i t_j = t_j s_{i+1} \\
t_i t_j = t_{i+1} t_{i+1}
\end{cases}
\end{align*}
\]

\[
\begin{array}{ccccccc}
\cdot & & & & & & \\
\downarrow & & & & & & \\
\cdot & & & & & & \\
\downarrow & & & & & & \\
\cdot & & & & & & \\
\end{array}
\]
The representable semi-cubical sets are the *semi-cubes*

\[ C^0 \rightarrow C^1 \rightarrow \cdots \rightarrow C^2 \rightarrow \cdots \rightarrow C^3 \]
The representable semi-cubical sets are the *semi-cubes*

\[
C^0 \cdot C^1 \rightarrow C^2 \downarrow \downarrow \\ \downarrow \downarrow \\
C^3
\]

Semi-cubical pasting schemes are rectangular grids

Example:
The representable semi-cubical sets are the *semi-cubes*

\[ \cdots C^0 \rightarrow C^1 \rightarrow C^2 \rightarrow C^3 \cdots \]

Semi-cubical pasting schemes are rectangular grids

Example:

A semi-cubical pasting scheme \( P \) of dimension \( n \) has \( n \) boundary \( \partial_i P \), with maps

\( \partial_i^- P, \partial_i^+ P : \partial_i P \rightarrow P \), for \( 0 \leq i < n \)
Define the *cubical sums*: diagrams for which pasting schemes are limits, and
*cubical extension*: a category $C$ equipped with a functor $\square \to C$ such that $C$ has
the cubical sums, with the initial cubical extension $\Theta_0^\square$. 
Define the *cubical sums*: diagrams for which pasting schemes are limits, and *cubical extension*: a category $C$ equipped with a functor $\square \to C$ such that $C$ has the cubical sums, with the initial cubical extension $\Theta_0$.

*Cubical theory*: a cubical extension $C$ such that the unique map $\Theta_0 \to C$ is faithful and identity on objects.

Intuition: a cubical theory $C$ contain pasting schemes with operations producing extra cells.
Define the *cubical sums*: diagrams for which pasting schemes are limits, and *cubical extension*: a category $C$ equipped with a functor $\square \to C$ such that $C$ has the cubical sums, with the initial cubical extension $\Theta_0^\square$.

*Cubical theory*: a cubical extension $C$ such that the unique map $\Theta_0^\square \to C$ is faithful and identity on objects.

Intuition: a cubical theory $C$ contain pasting schemes with operations producing extra cells.

A family of cells $x_1, \ldots, x_n$ in an object $P$ of a cubical theory are simultaneously algebraic if there are no non-trivial map $f : Q \to P$ in $\Theta_0^\square$ such that all the $x_i$ are in the image of $f$. 


A family of cells \((x_1, \ldots, x_n, y_1, \ldots, y_n)\) of dimension \(n - 1\) is \textit{compatible} if the cells fit in the boundary of an \(n\)-cube. A lift of a family of compatible cells \(x_1, \ldots, x_n, y_1, \ldots, y_n\) of dimension \(n - 1\) is a cell \(z\) of dimension \(n\) is a cell \(z\) such that \(s_i(z) = x_i\) and \(t_i(z) = y_i\).
Coherator for Semi-Cubical Weak $\omega$-categories

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- The coherator $\Theta_\square^\infty$ is the cubical theory constructed as follows

$$\Theta_\square^\infty = \lim(\Theta_0^\square \rightarrow \Theta_1^\square \rightarrow \Theta_2^\square \rightarrow \ldots)$$

where $\Theta_{n+1}^\square$ is formally obtained from $\Theta_n^\square$ by universally adding a lift for every compatible family of cells $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ in $P$, where either:

- we can decompose $x_i = \partial_{i,P}^- (x'_i)$ and $y_i = \partial_{i,P}^+ (y'_i)$ with $x'_i, y'_i$ algebraic in $\partial_i P$
- both families $(x_i)$ and $(y_i)$ are algebraic in $P$

and for which a lift was not added at an earlier stage.
Semi-cubical weak $\omega$-categories are presheaves over $\Theta_\infty$ that preserve the cubical sums.
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Consider a cubical weak $\omega$-category $X$:

- For every 0-cell $x$, we can construct a 1-cell $\text{id}(x) : x \to x$
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- For every diagram $x \xrightarrow{f} y \xrightarrow{g} z$, we can construct a 1-cell $f \star_0 g : x \to z$
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Consider a cubical weak $\omega$-category $X$:

- For every 0-cell $x$, we can construct a 1-cell $\text{id}(x) : x \to x$

- For every diagram $x \xrightarrow{f} y \xrightarrow{g} z$, we can construct a 1-cell $f \star_0 g : x \to z$

- For every diagram $x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} w$, we can construct a 2-cell $f \star_0 g$.
For every diagram

\[
\begin{align*}
  x & \xrightarrow{f} y \\
  h & \downarrow \alpha \downarrow k \\
  x' & \xrightarrow{f'} y' \\
  h' & \downarrow \alpha' \downarrow k' \\
  x'' & \xrightarrow{f''} y''
\end{align*}
\]

we have a 2-cell

\[
\begin{align*}
  x & \xrightarrow{f} y \\
  h*0h' & \downarrow \alpha*1\alpha' \downarrow k*0k' \\
  x'' & \xrightarrow{f''} y''
\end{align*}
\]

\[
\Rightarrow \quad \begin{align*}
  (\alpha*1\alpha') & \xrightarrow{1} (\alpha*0\beta) \xrightarrow{1} (\alpha'\beta')
\end{align*}
\]
For every diagram \( x \xrightarrow{f} y \), we have a 2-cell \( h \downarrow \downarrow \alpha \downarrow k \).

For every diagram \( x' \xrightarrow{f'} y' \), we have a 2-cell \( h' \downarrow \downarrow \alpha' \downarrow k' \).

For every diagram \( x'' \xrightarrow{f''} y'' \), we have a 2-cell \( f'' \downarrow \downarrow k'' \).

For every diagram \( x \xrightarrow{f} y \xrightarrow{g} z \), we have a 2-cell \( h \downarrow \downarrow \alpha \downarrow k \downarrow \beta \downarrow l \).

For every diagram \( x \xrightarrow{f \ast 0 g} z \), we have a 2-cell \( h \downarrow \downarrow \alpha \downarrow k \downarrow \beta \downarrow l \).

For every diagram \( x' \xrightarrow{f'} y' \xrightarrow{g'} z' \), we have a 2-cell \( f' \downarrow \downarrow \beta \downarrow l \).

For every diagram \( x \xrightarrow{f \ast 0 g} z \), we have a 2-cell \( h \downarrow \downarrow \alpha \downarrow k \downarrow \beta \downarrow l \).
For every diagram
\[ x \xrightarrow{f} y \xrightarrow{h} \xrightarrow{\downarrow \alpha} \xrightarrow{\downarrow k} \]
\[ x' \xrightarrow{f'} y' \xrightarrow{h'} \xrightarrow{\downarrow \alpha'} \xrightarrow{\downarrow k'} \]
\[ x'' \xrightarrow{f''} y'' \]
we have a 2-cell
\[ x \xrightarrow{f} y \xrightarrow{h*0h'} \xrightarrow{\downarrow \alpha*1\alpha'} \xrightarrow{\downarrow k*0k'} \]
\[ x'' \xrightarrow{f''} y'' \]

For every diagram
\[ x \xrightarrow{f} y \xrightarrow{g} \xrightarrow{h} \xrightarrow{\downarrow \alpha} \xrightarrow{\downarrow k} \xrightarrow{\downarrow \beta} \xrightarrow{\downarrow l} \]
\[ x' \xrightarrow{f'} y' \xrightarrow{g'} \xrightarrow{h'} \xrightarrow{\downarrow \alpha'} \xrightarrow{\downarrow k'} \xrightarrow{\downarrow \beta'} \xrightarrow{\downarrow l'} \]
\[ x'' \xrightarrow{f''} y'' \xrightarrow{g''} \xrightarrow{h''} \xrightarrow{\downarrow \alpha''} \xrightarrow{\downarrow k''} \xrightarrow{\downarrow \beta''} \xrightarrow{\downarrow l''} \]
gives
\[ (\alpha*_{1} \alpha')*_{0} (\beta*_{1} \beta') \Rightarrow (\alpha*_{0} \beta)*_{1} (\alpha'*_{0} \beta') \]
Interesting Note on Weak Degeneracies

For every 1-cell $x \xrightarrow{f} y$, one can construct two identity 2-cells

\[
\begin{array}{ccc}
  x & \xrightarrow{f} & y \\
  \downarrow \text{id}(x) & & \downarrow \text{id}(y) \\
  x & \xrightarrow{f} & y \\
\end{array}
\]

\[
\begin{array}{ccc}
  x & \xrightarrow{\text{id}(x)} & x \\
  \downarrow \text{id}_0(f) & & \downarrow f \\
  y & \xrightarrow{\text{id}(y)} & y \\
\end{array}
\]

and $\text{id}_1(f)$ have the same type, and are equivalent, but not strictly equal!
For every 1-cell $x \xrightarrow{f} y$, one can construct two identity 2-cells

$id_1(id(x))$ and $id_0(id(x))$ have the same type, and are equivalent, but not strictly equal!
Thank you!