The smothering model structure and prederivators as $(\infty, 1)$-categories

Work in progress

El Mehdi Cherradi

IRIF - CNRS - Université Paris Cité, MINES ParisTech - Université PSL

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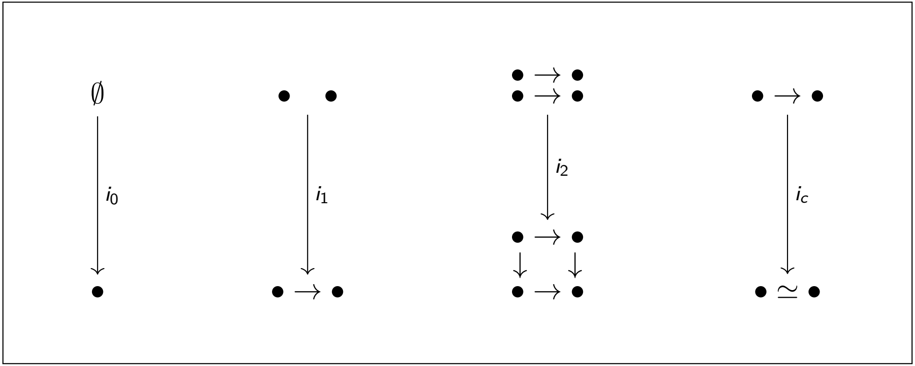
Outline

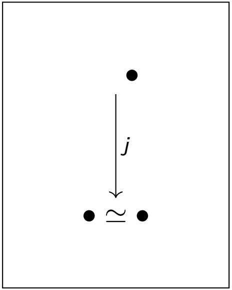
The generating cofibrations

We define a functor to be **(stably) smothering** if it has the RLP with respect to the following inclusions:



The generating trivial cofibration

The fibrations are just the **isofibrations**, which can be defined as having the RLP with respect to the inclusion j pictured on the right.



The model structure

Proposition

There exists a model structure on **Cat** *with:*

- *fibrations the isofibrations*
- *trivial fibrations the (stably) smothering functors*
- *weak equivalences the weakly (stably) smothering functors*

Hence, it is a right Bousfield localization of the natural model structure on **Cat** that proves to be relevant in settings involving homotopy categories.

Example

Proposition

Consider a pullback squares of quasicategories

$$\begin{array}{ccc} A \times_B E & \longrightarrow & E \\ \big\downarrow & \lrcorner & \big\downarrow p \\ A & \xrightarrow[f]{} & B \end{array}$$

where p is an isofibration.
Then the canonical functor

$$Ho(A \times_B E) \to Ho(A) \times_{Ho(B)} Ho(E)$$

is a weakly smothering.

Example

Proposition

For $\mathcal{M}$ a model category, the canonical functors

$$\mathbf{Ho}(\mathcal{M}^{\to\to}) \to \mathbf{Ho}(\mathcal{M}^{\to}) \times_{\mathbf{Ho}(\mathcal{M})} \mathbf{Ho}(\mathcal{M}^{\to})$$

and

$$\mathbf{Ho}(\mathcal{M}^{\to}) \to \mathbf{Ho}(\mathcal{M})^{\to}$$

are weakly smothering.

Outline

Prederivators

Recall that a prederivator is a (strict) 2-functor from $\mathbf{Cat}^{op}$ (small categories) to $\mathbf{CAT}$ (large categories).

Example

For $\mathcal{M}$ a model category, the 2-functor

$$\begin{aligned} \mathbb{D} : \mathbf{Cat}^{op} &\to \mathbf{CAT} \\ J &\mapsto \mathbf{Ho}(\mathcal{M}^J) \end{aligned}$$

is a prederivator

The transfered Reedy model structure

Proposition

There exists a right transfered model structure along the restriction functor

$$\mathbf{PDer}(= \mathbf{CAT}^{\mathbf{Cat}^{op}}) \to \mathbf{CAT}^{\Delta^{op}}$$

where Δ^{op} is seen as a discrete 2-category.

Complete Segal prederivator

Definition

A "Reedy fibrant" prederivator $\mathbb{D}$ is defined to be a Segal prederivator if the canonical functors

$$\mathbb{D}([n]) \to \mathbb{D}([1]) \times_{\mathbb{D}([0])} \dots \times_{\mathbb{D}([0])} \mathbb{D}([1])$$

is weakly smothering for every $n \geq 2$.
If the canonical decoherence functor

$$\mathbb{D}([1]) \to \mathbb{D}([0])^{\to}$$

is also weakly smothering, we will say that $\mathbb{D}$ is complete.

The model structure

Proposition

There exists a left Bousfield localization of the transfered Reedy model structure on **PDer** *whose fibrant objects are the complete Segal prederivators.*

Theorem

The functor

$$\begin{aligned} \mathbf{Ob} : \mathbf{PDer}_{CSP} &\to \mathbf{SSet}_{Joyal} \\ \mathbb{D} &\mapsto S_n := \mathbf{Ob}(\mathbb{D}([n])) \end{aligned}$$

is right Quillen and the induced right derived functor is an equivalence of categories, thus providing a Quillen equivalence.

Thank you for you attention!

References

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