LIFTING & LENSES

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LHC days 2023

IRIF, Université Paris Cité

OVERVIEW & MOTIVATION

algebraic weak factorisation systems

GENERALISE

orthogonal factorisation systems

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- compatible comonad L and monad R on $\mathcal{C}^2$

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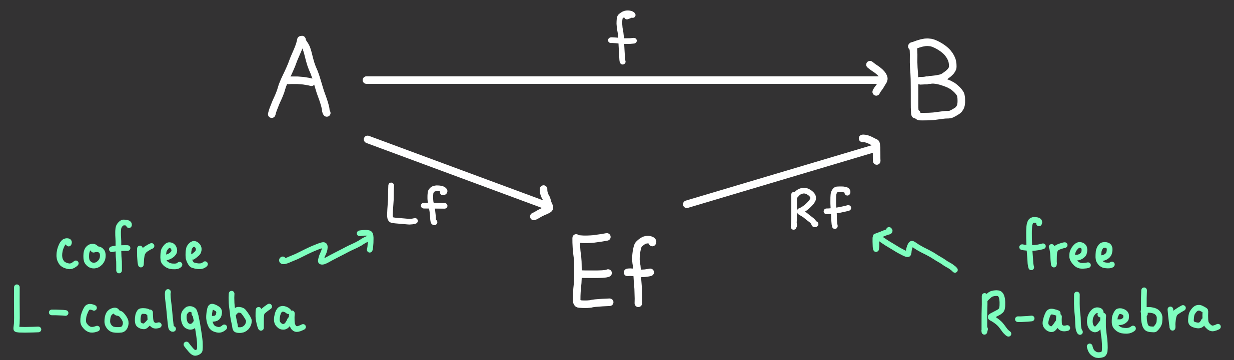

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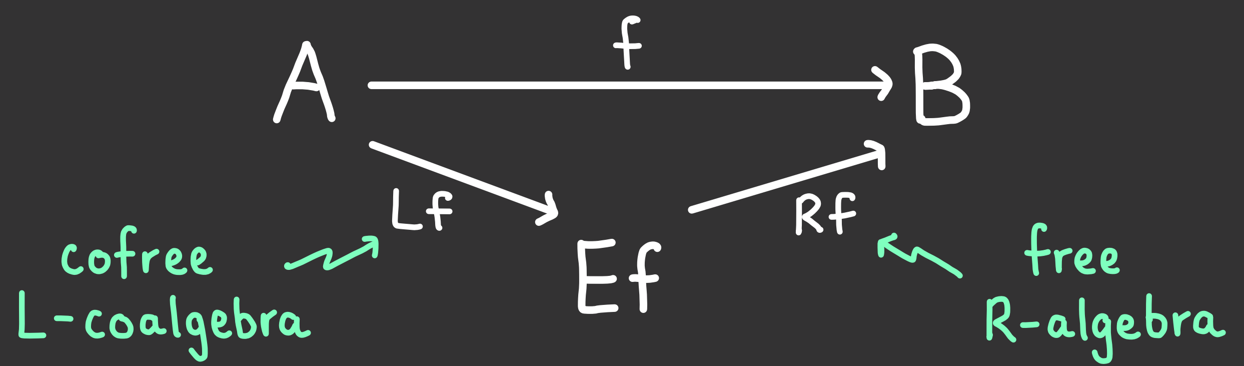
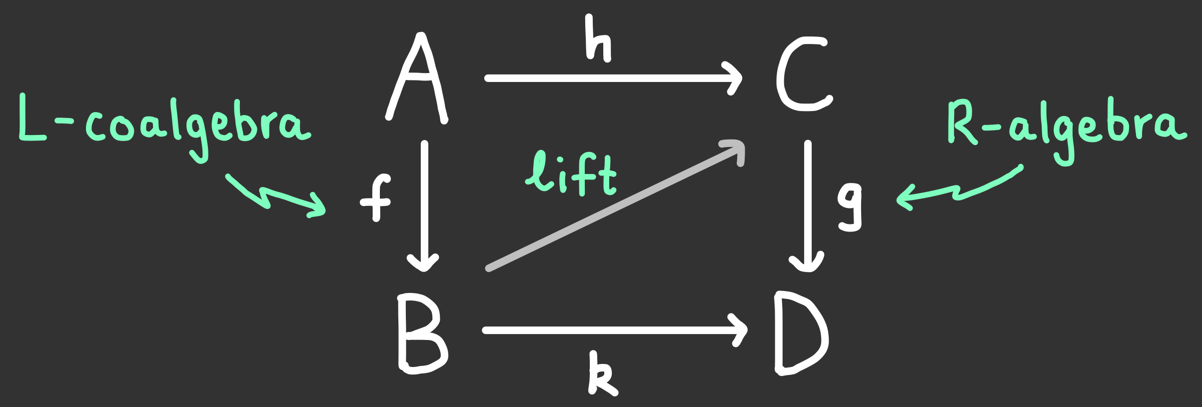


- lifts of L-coalgebras against R-algebras



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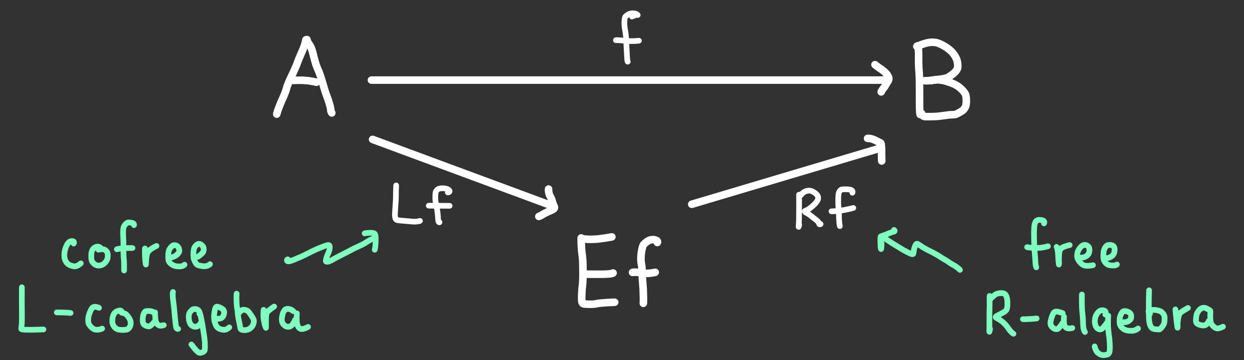

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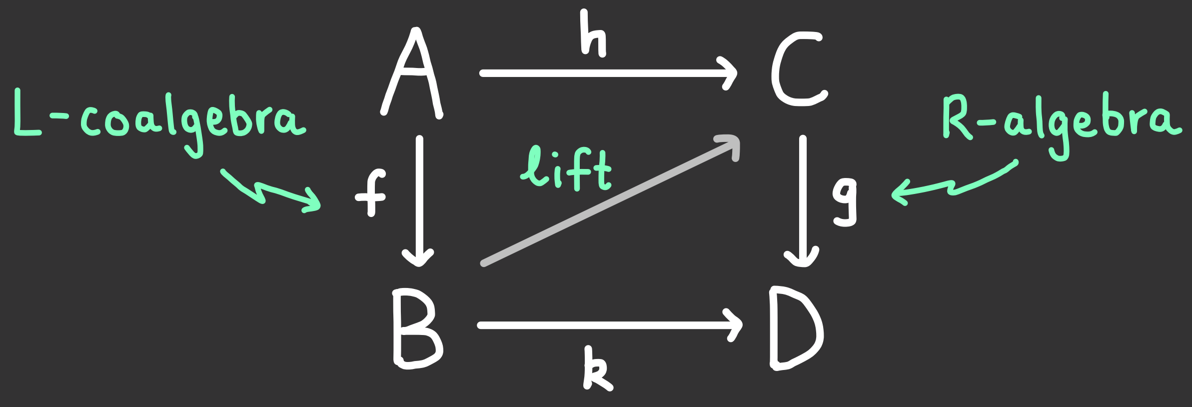

This talk: 2 related examples of AWFS

L-coalgebra	R-algebra
lari	split opfibration
???	delta lens

SPLIT OPFIBRATIONS

A split opfibration is a functor equipped with a lifting operation (splitting)

$$\begin{array}{ccc} A & \quad & a \xrightarrow{\varphi(a,u)} a' \\ {\scriptstyle f}\downarrow & & \\ B & & fa \xrightarrow{\quad u \quad} b \end{array}$$

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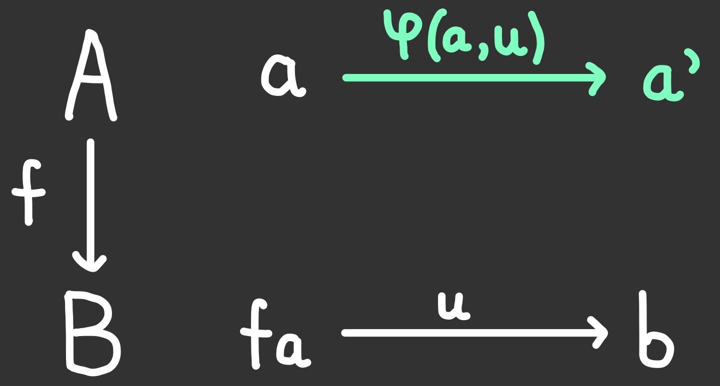

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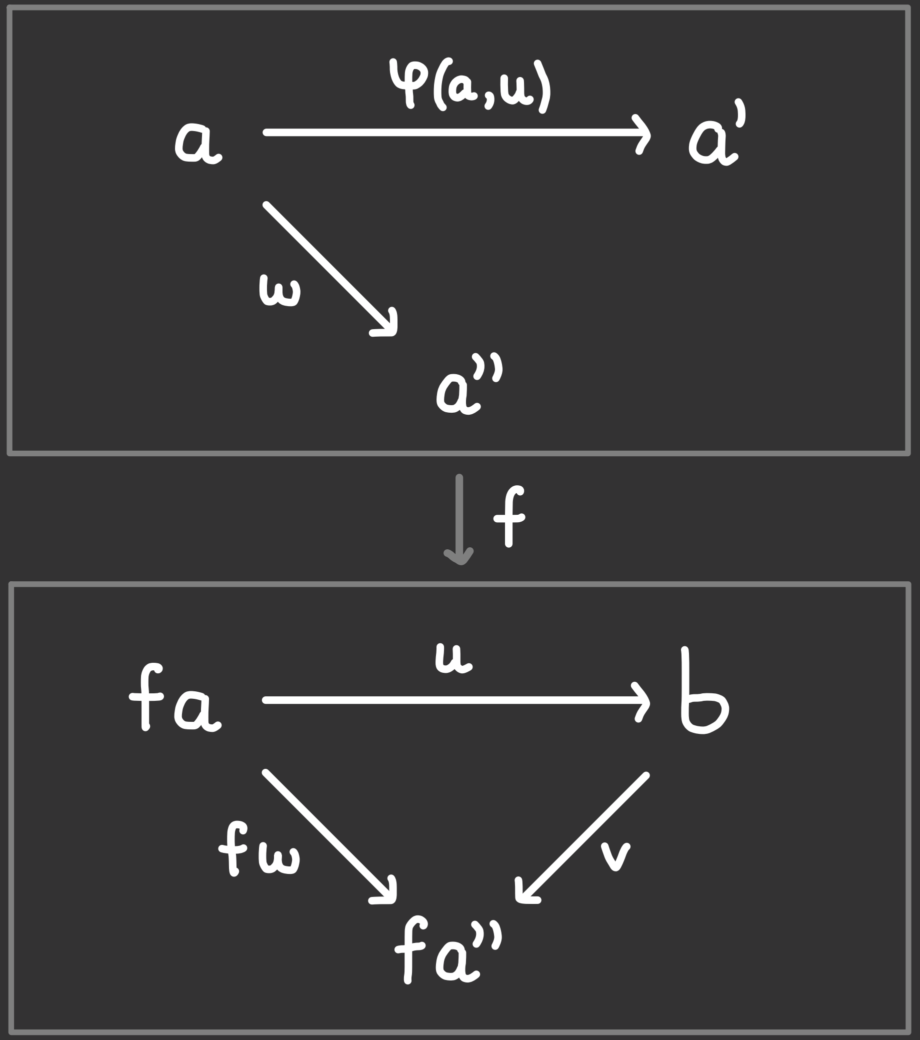

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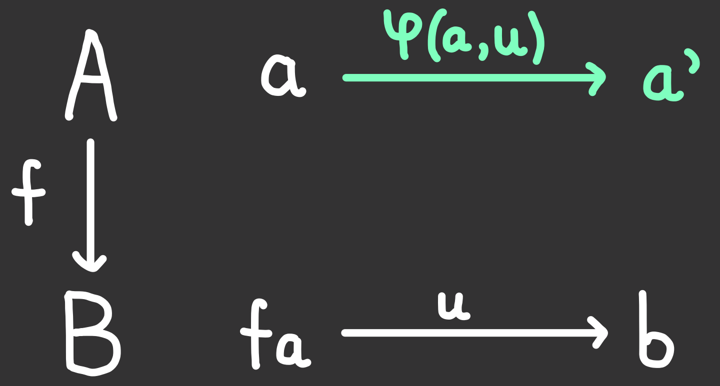

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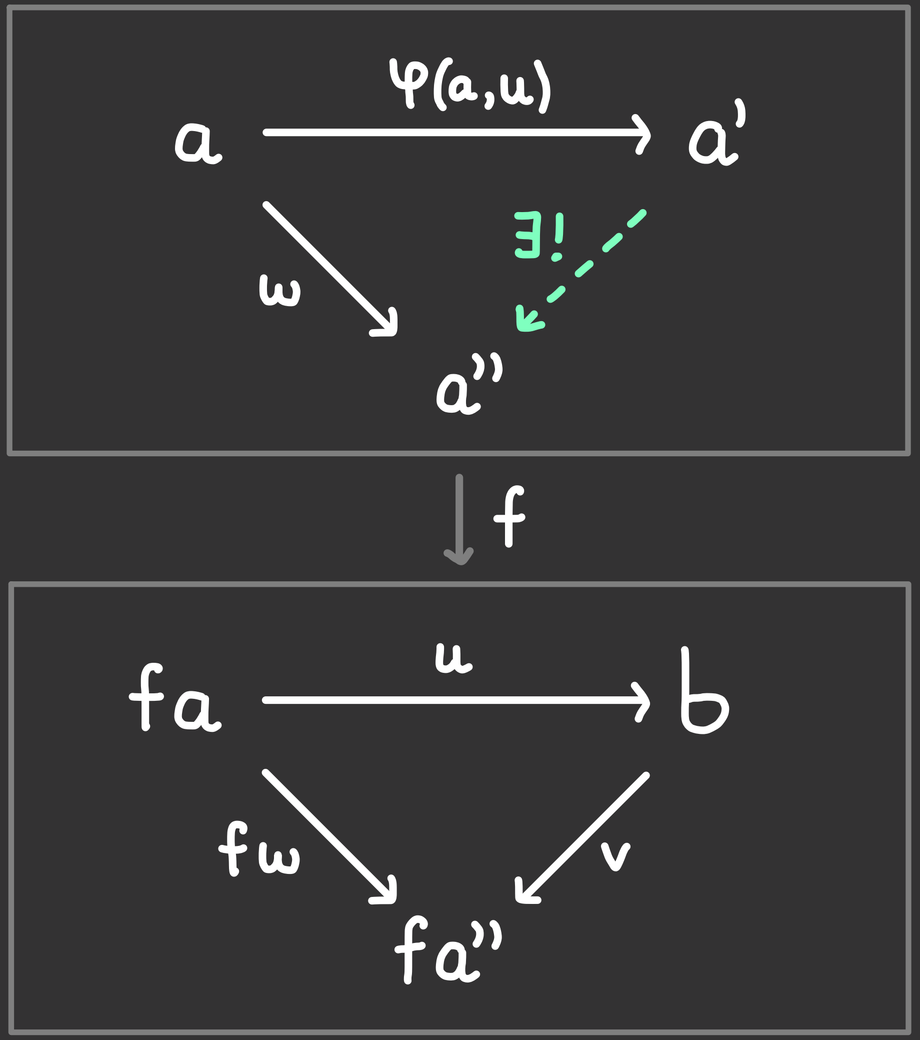

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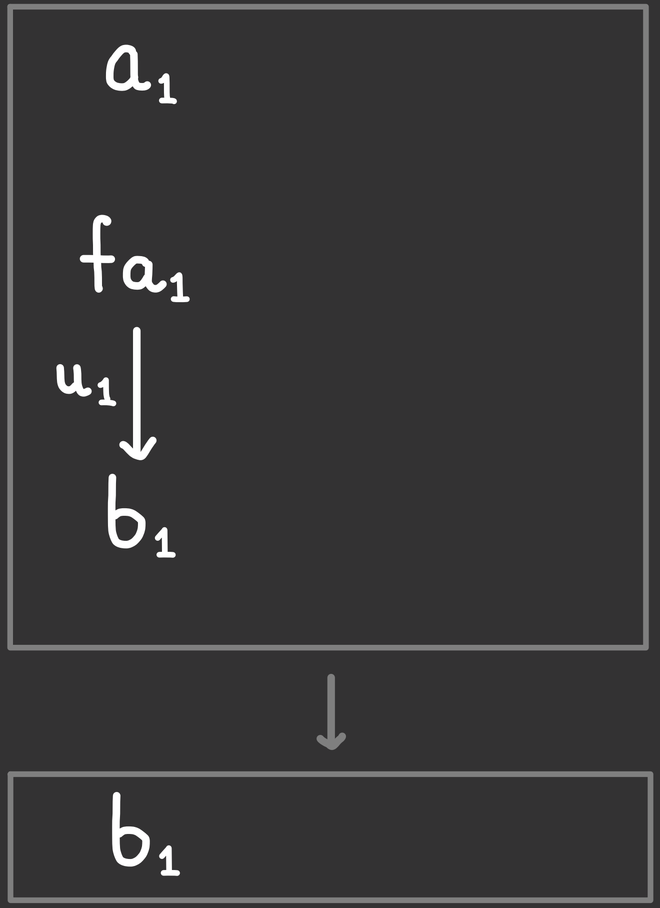

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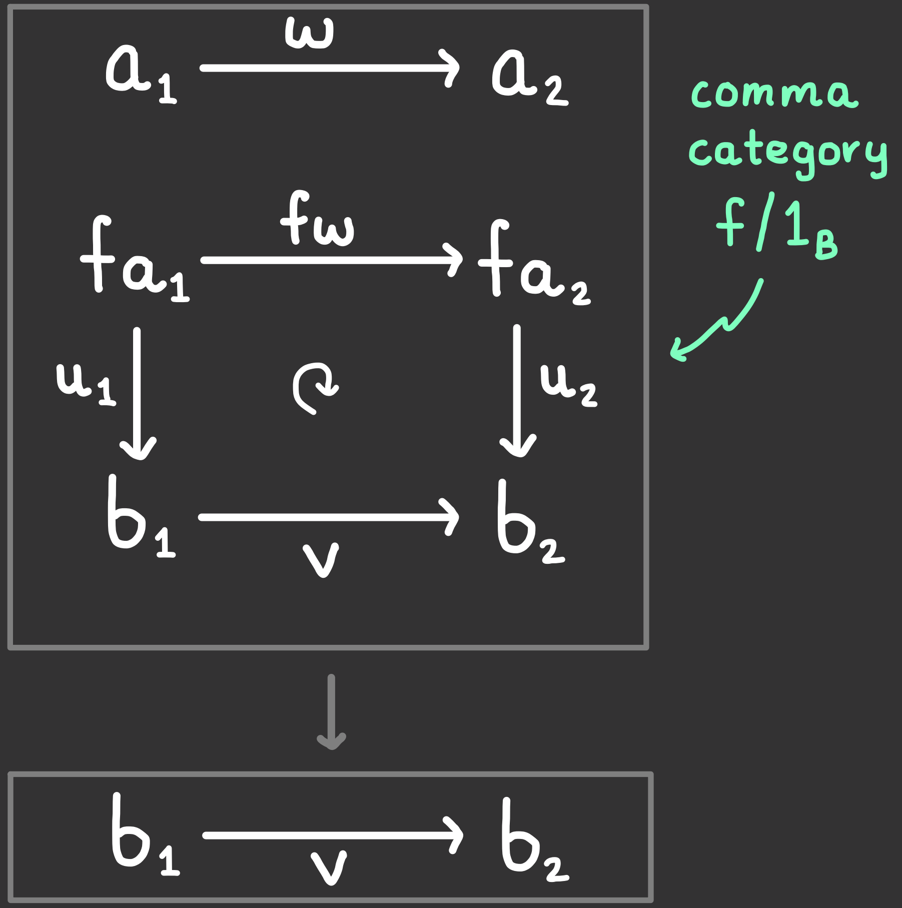

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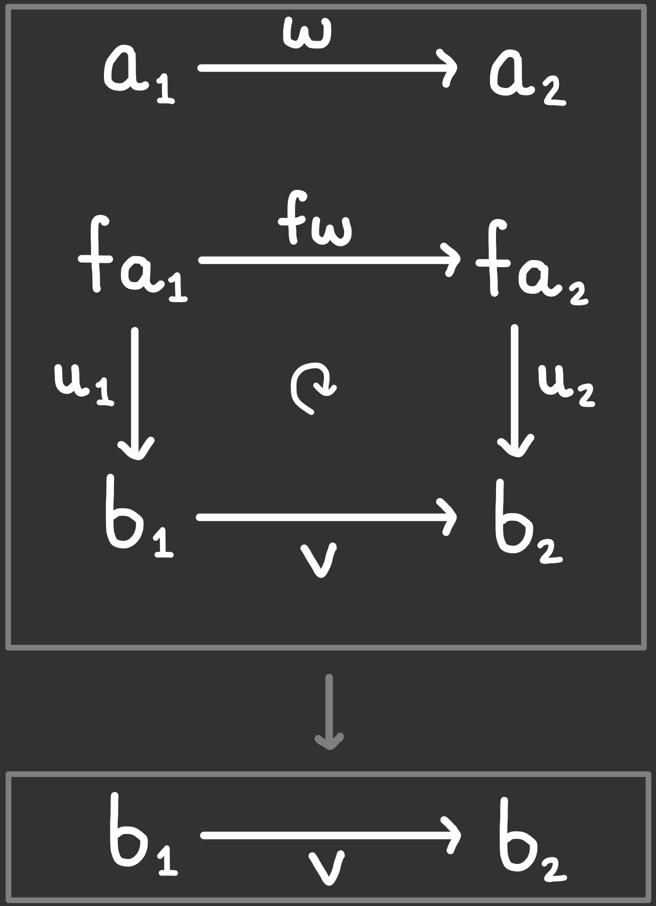

comma category $f/1_B$

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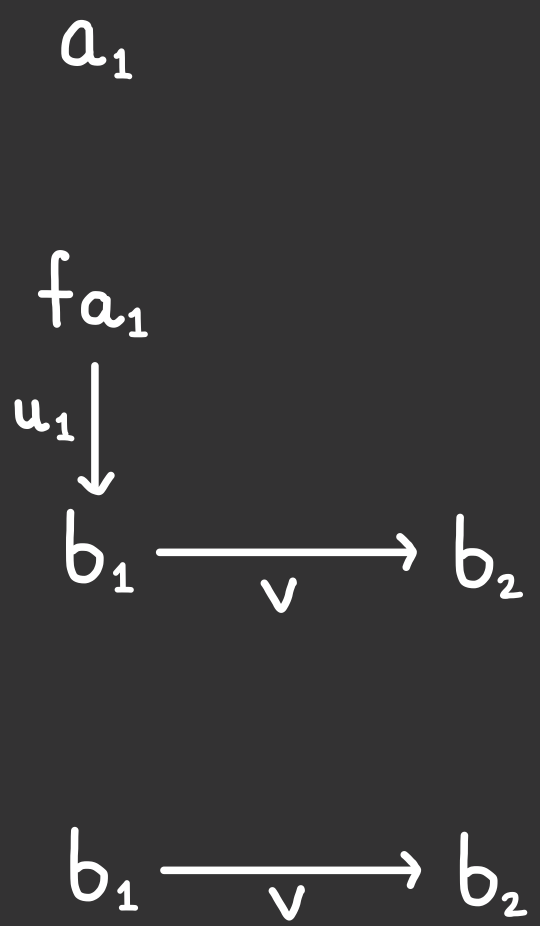

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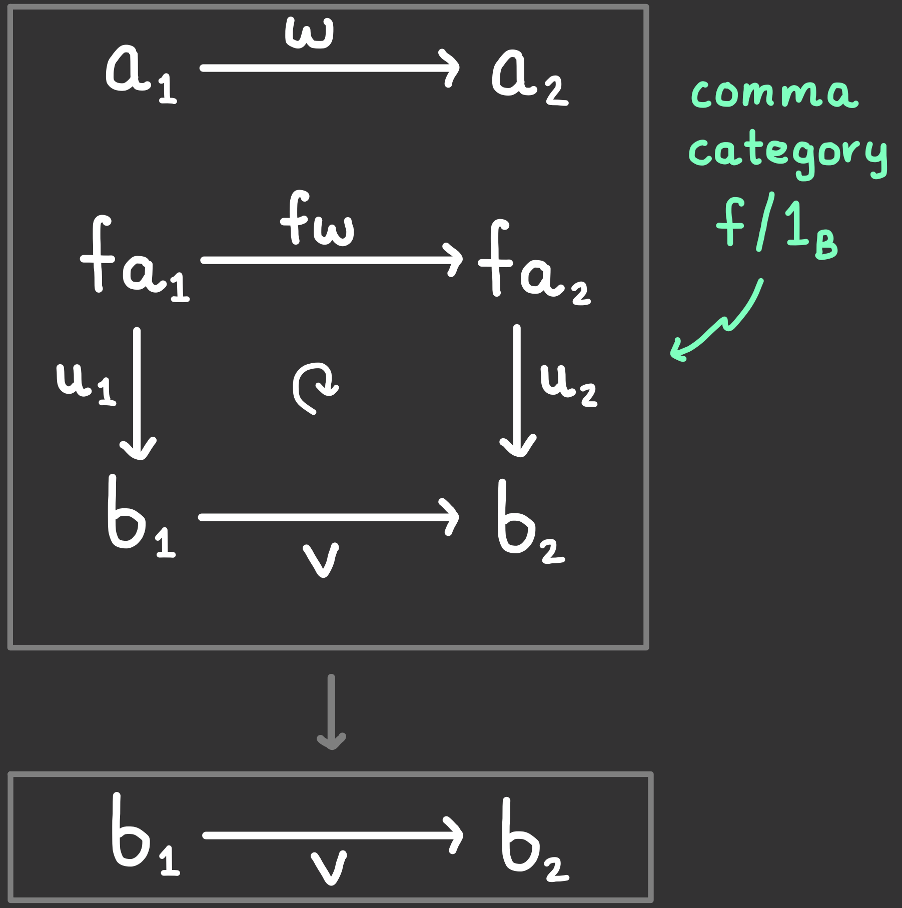

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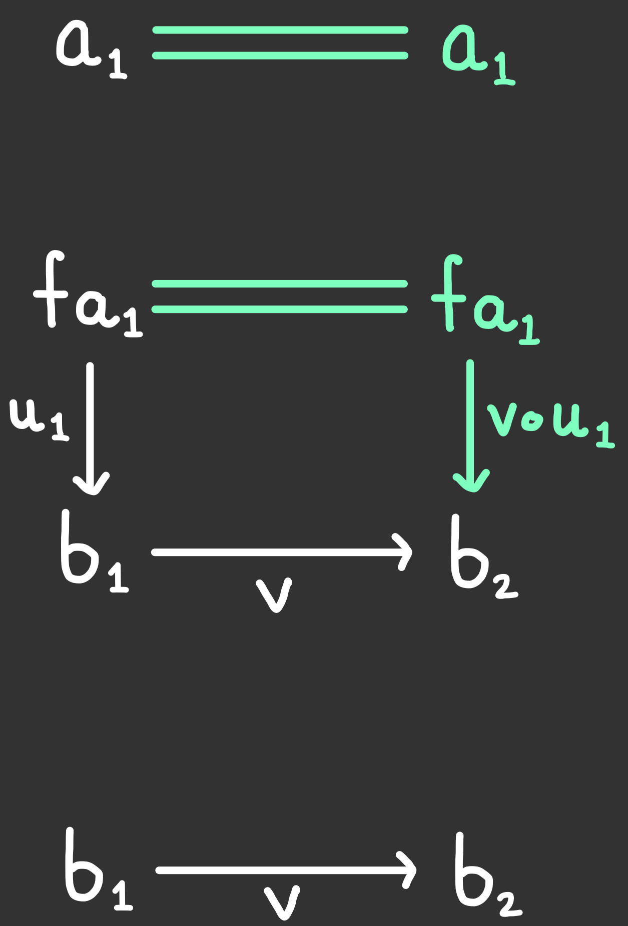

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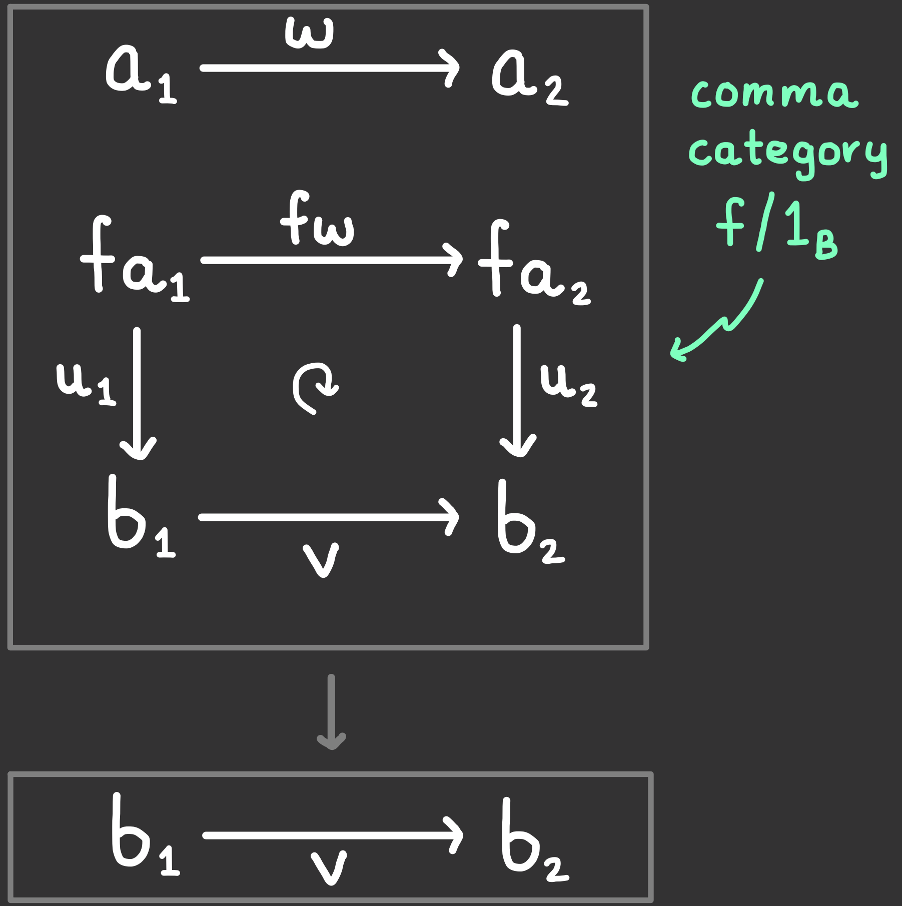

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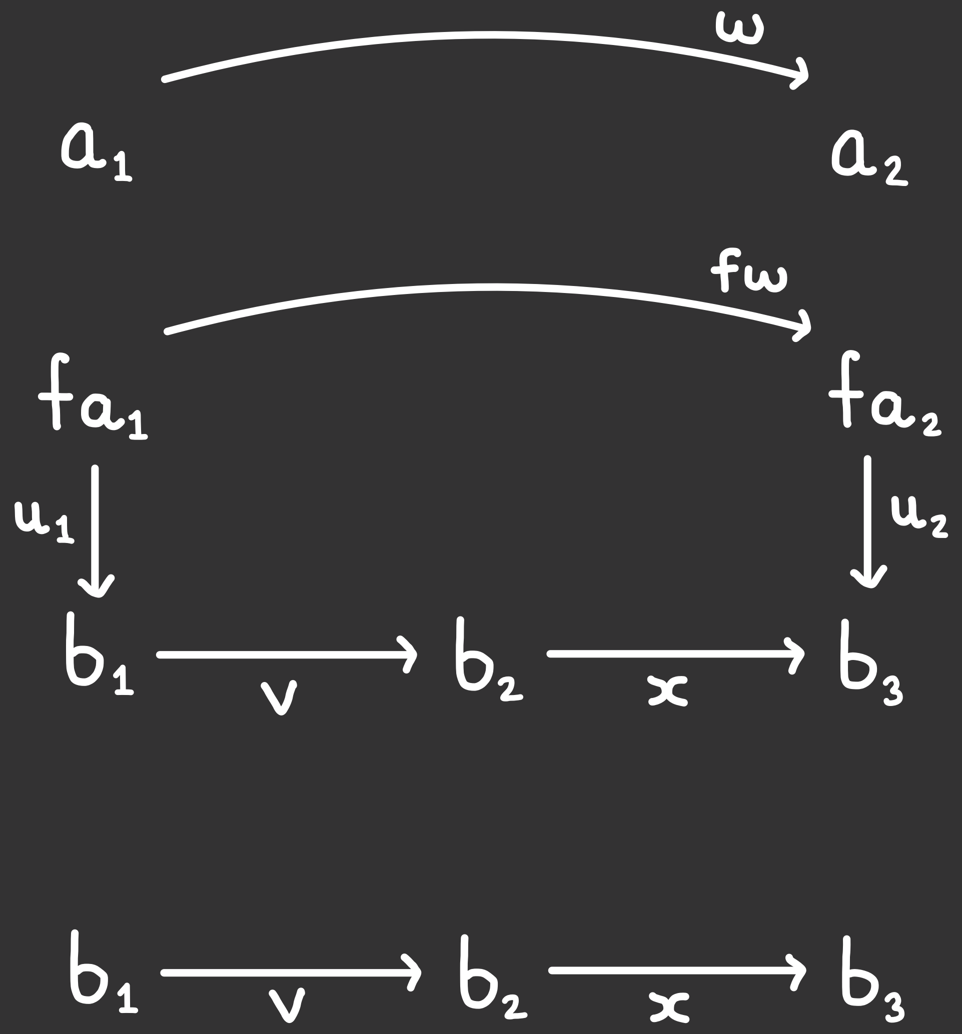

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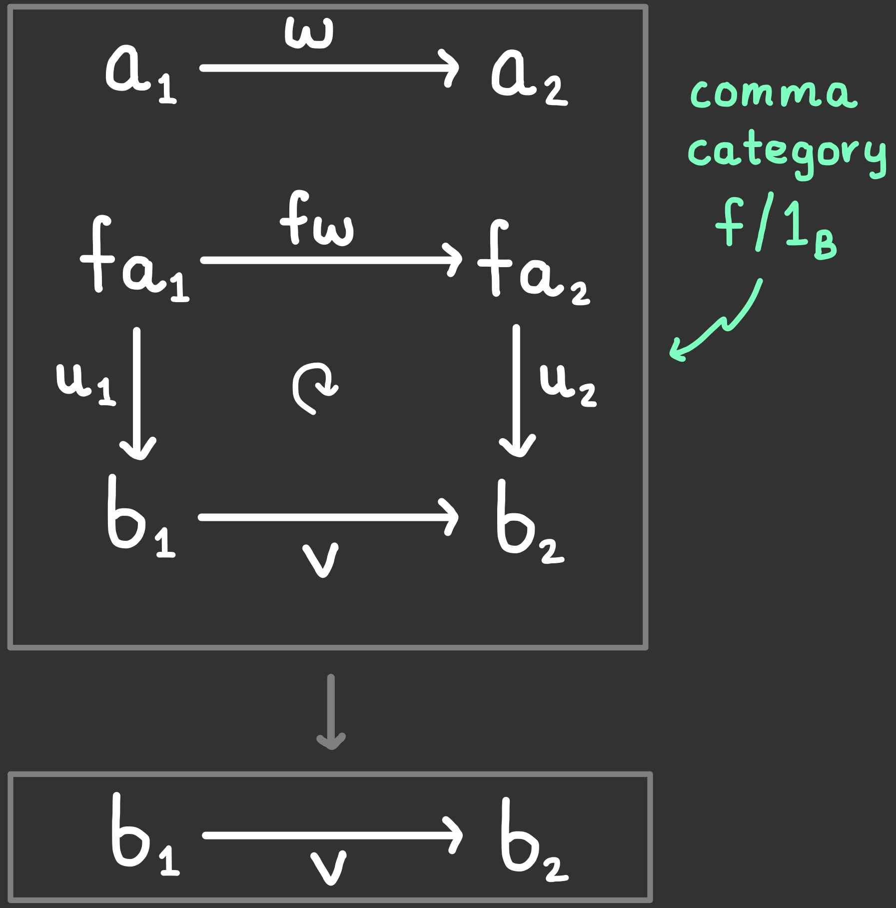

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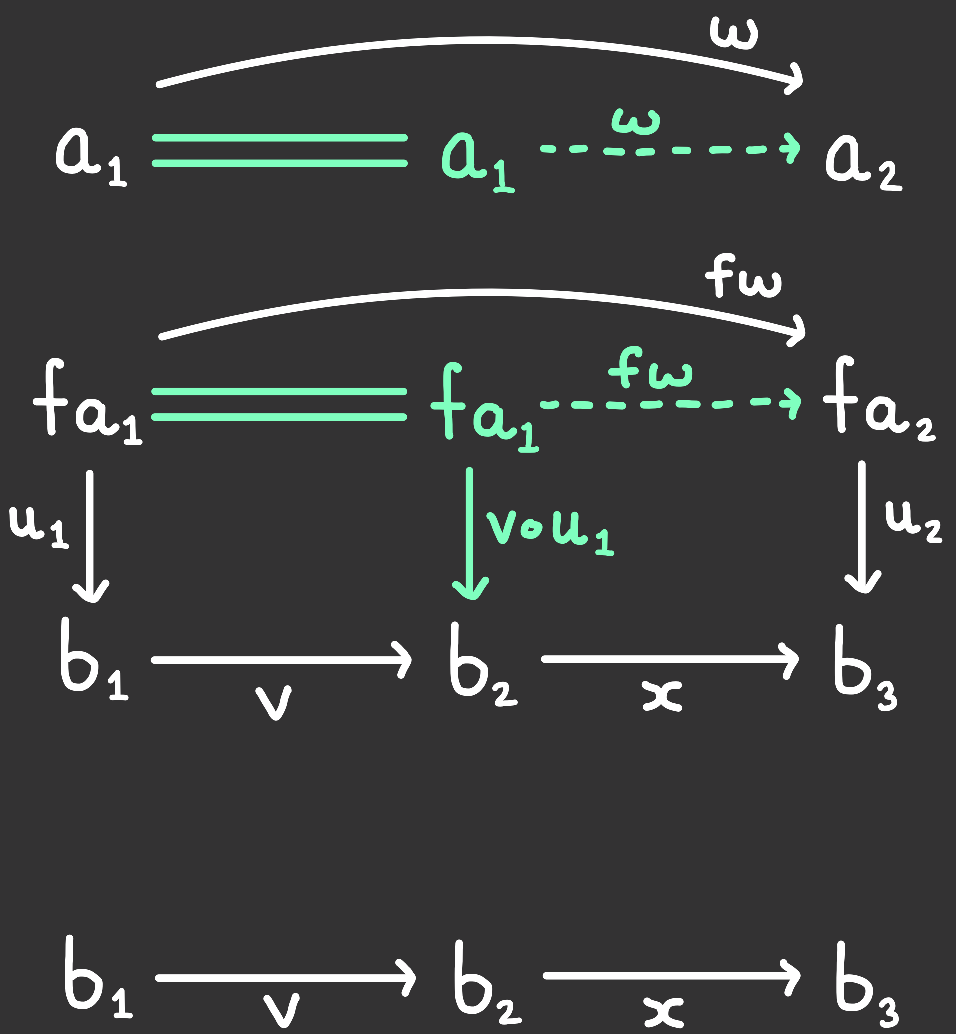
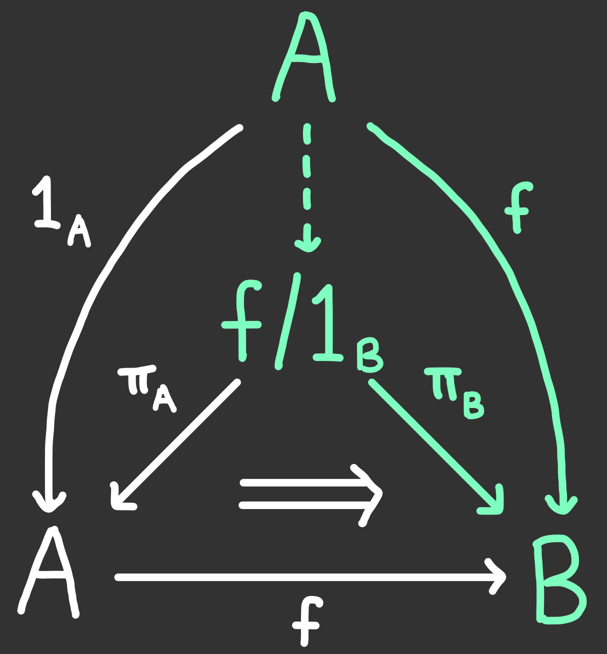


THE AWFS FOR SPLIT OPFIBRATIONS

Factorisation

Every functor factorises into a (cofree) left-adjoint-right inverse followed by a (free) split opfibration.



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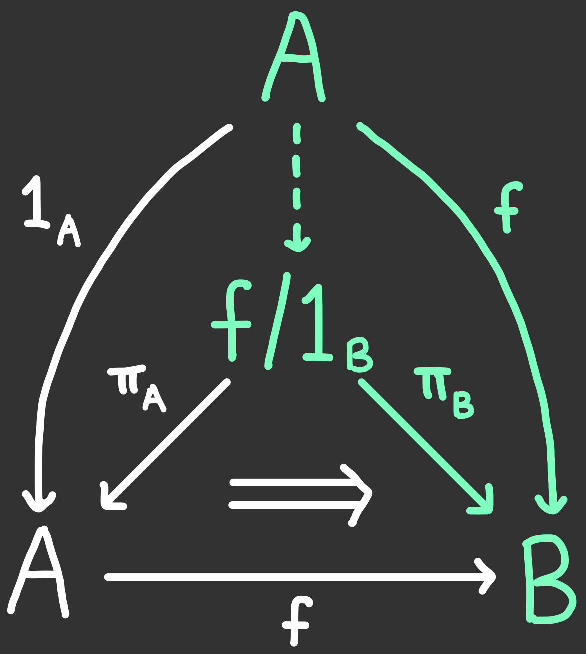
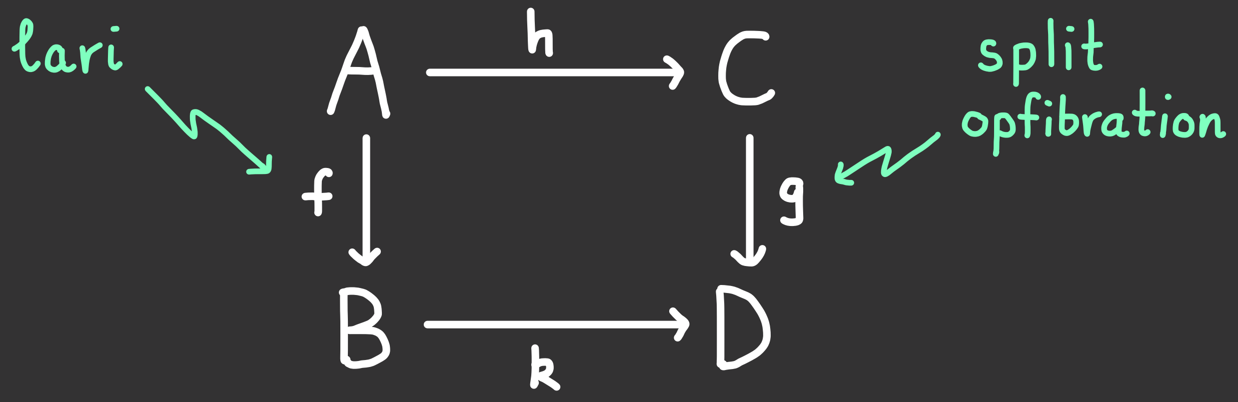


Lifting

Given a commutative square in Cat



there is a canonical functor $j : B \longrightarrow C$ such that $jf = h$ and $gj = k$.

E.g. Take $f : \{0 \longrightarrow 2\} \longrightarrow \{0 \longrightarrow 1 \longrightarrow 2\}$.

DELTA LENSES

A **delta lens** is a functor equipped with a lifting operation

$$A \qquad a \xrightarrow{\varphi(a,u)} a'$$

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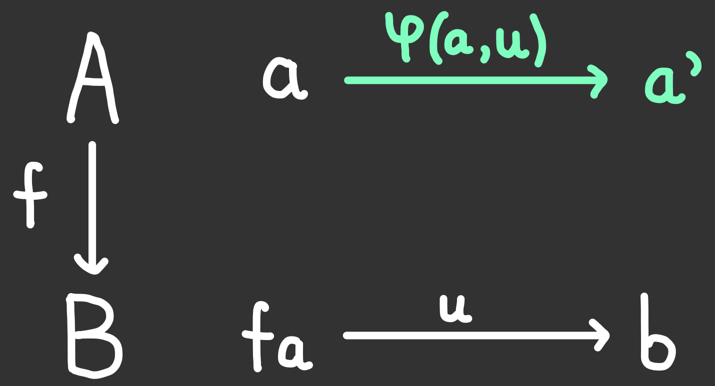

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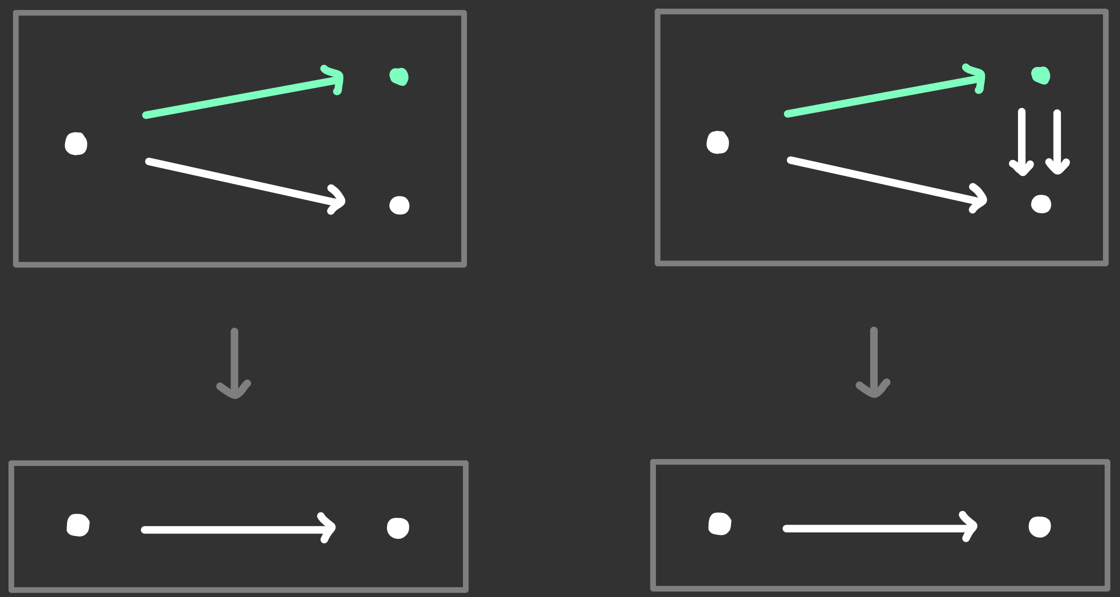
Two simple examples of delta lenses which are <u>not</u> split opfibrations.



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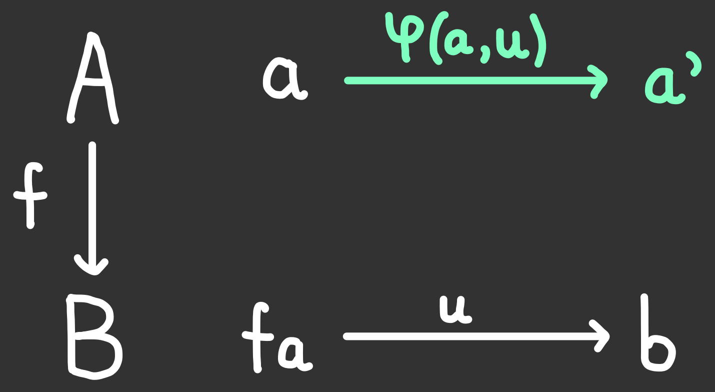

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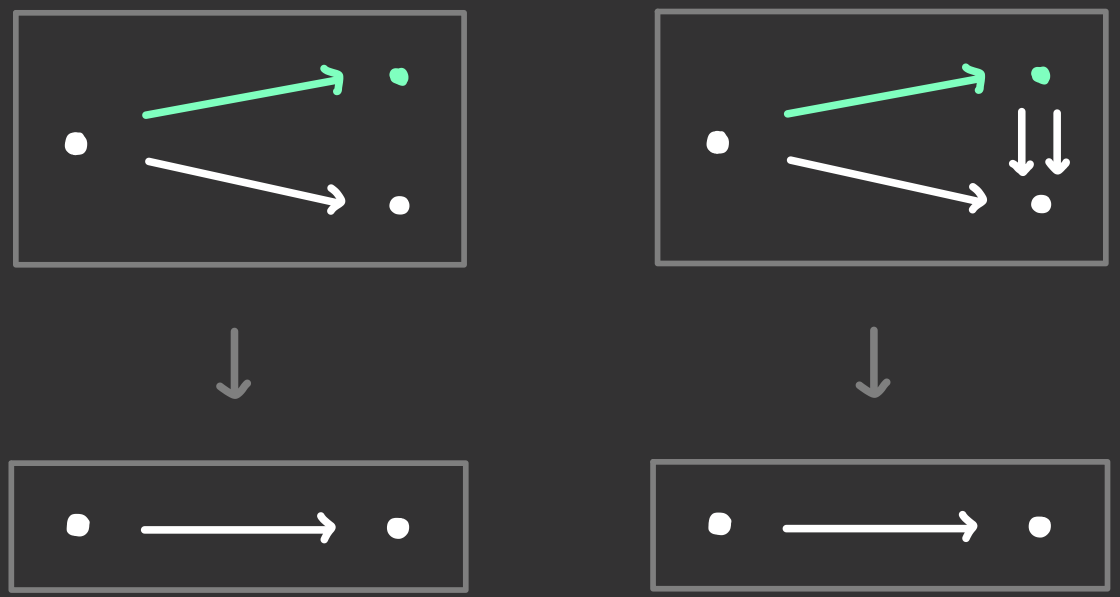

Q: What is the free delta lens ?

FREE DELTA LENSES (1)

The **free delta lens** $Rf : Ef \longrightarrow B$ on a functor $f : A \longrightarrow B$ has domain whose:

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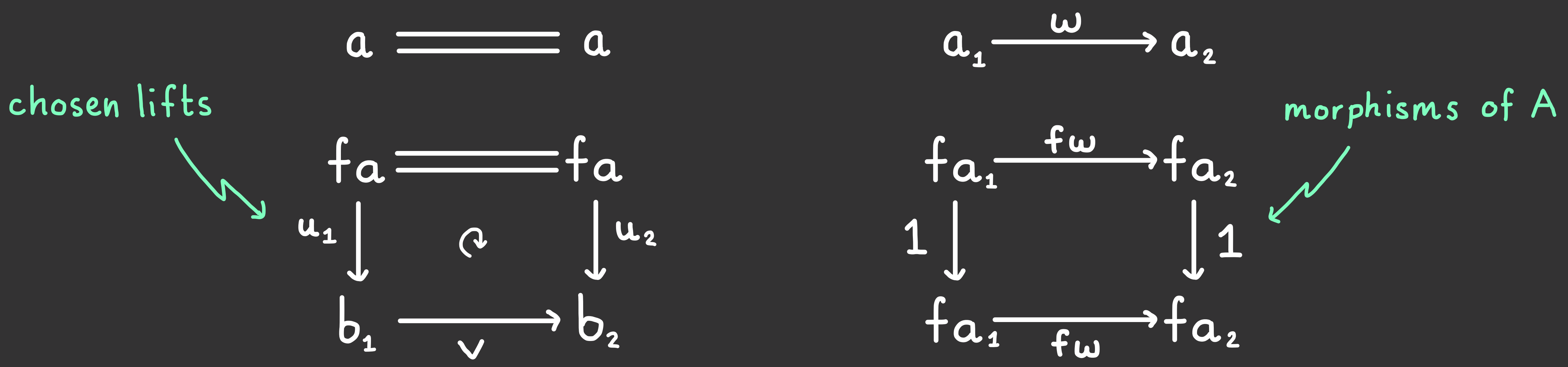

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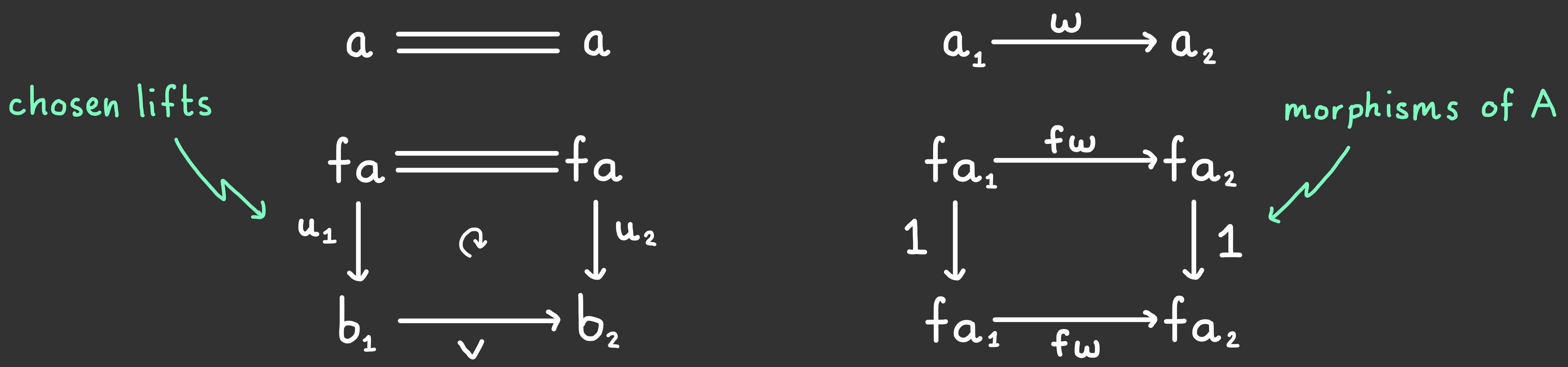

The functor Rf sends these generators to $v : b_1 \longrightarrow b_2$ and $fw : fa_1 \longrightarrow fa_2$, respectively.

FREE DELTA LENSES (2)

The free delta lens $Rf : Ef \longrightarrow B$ on a functor $f : A \longrightarrow B$ has domain whose:

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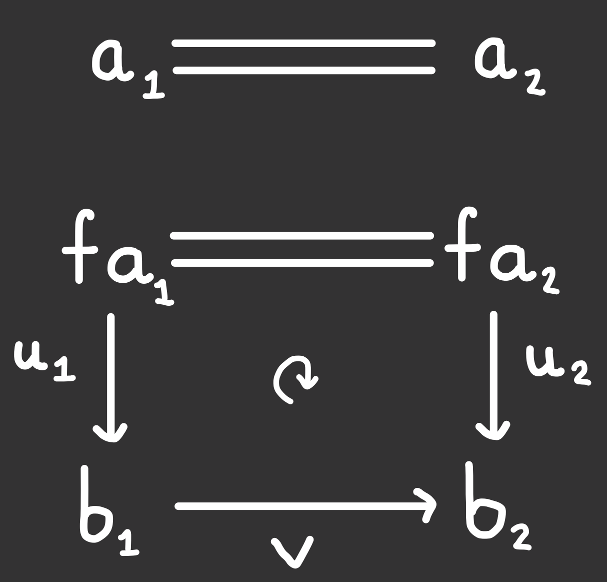
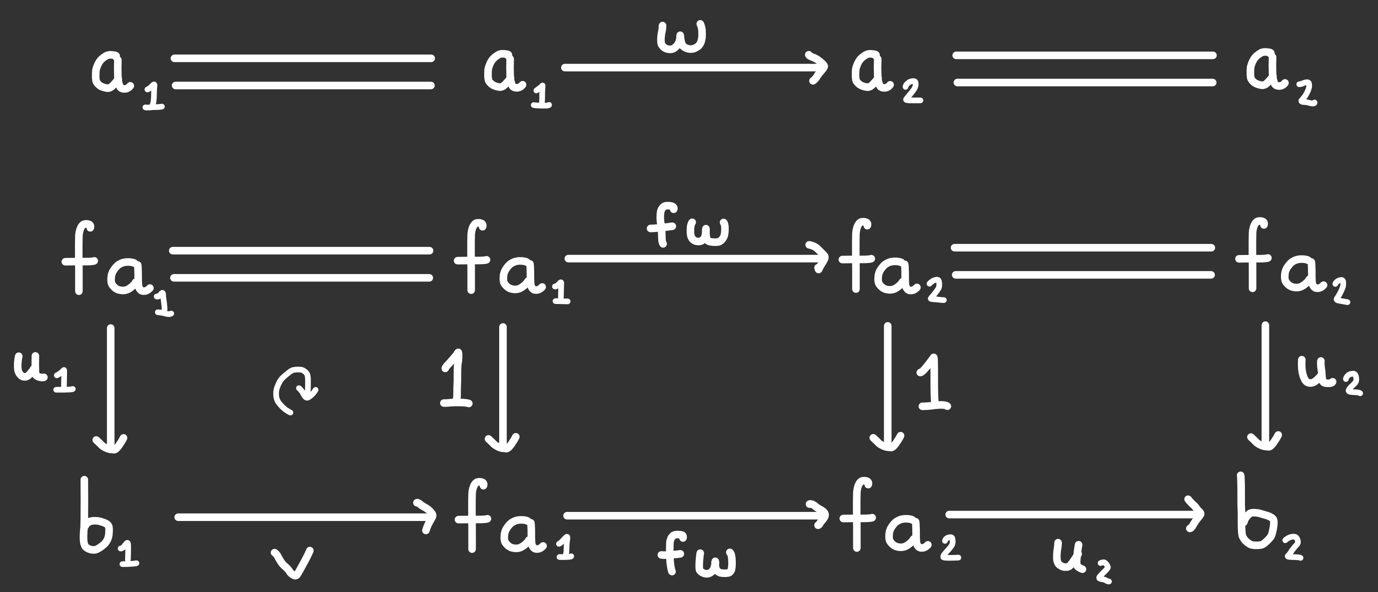
- morphisms $(a_1, u_1) \longrightarrow (a_2, u_2)$ are given by the following two sorts:





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THE AWFS FOR DELTA LENSES

Factorisation

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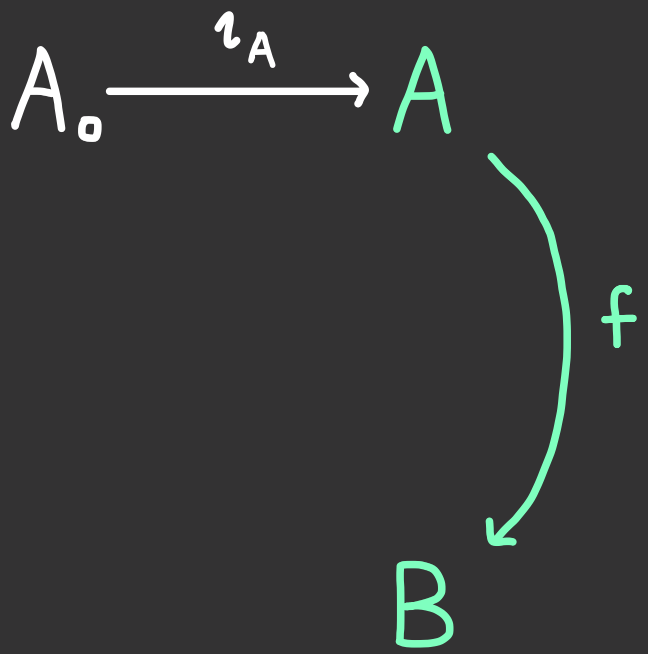

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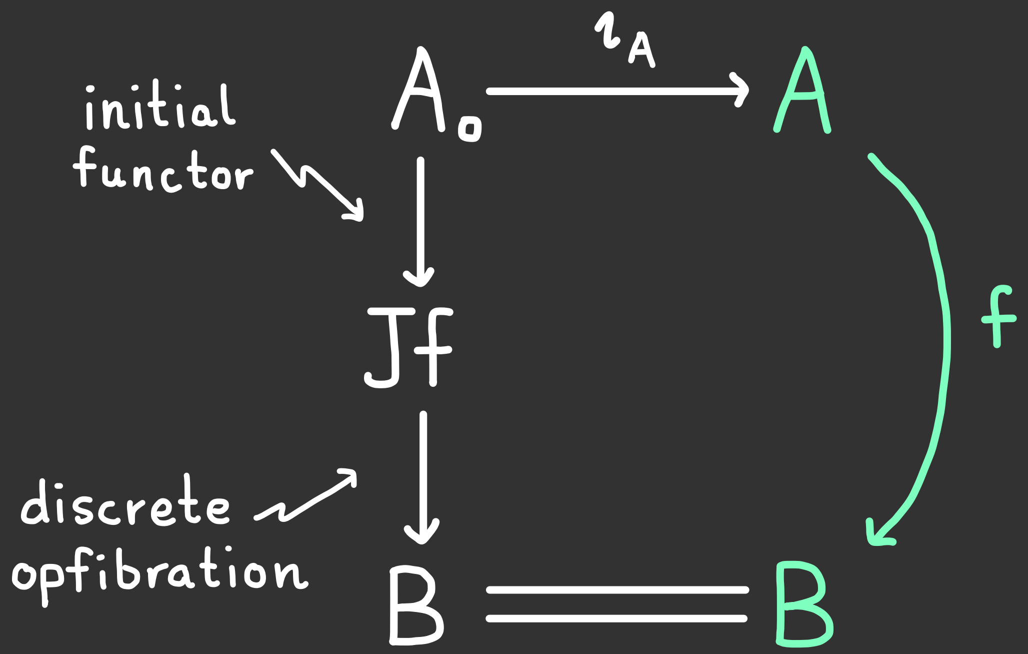

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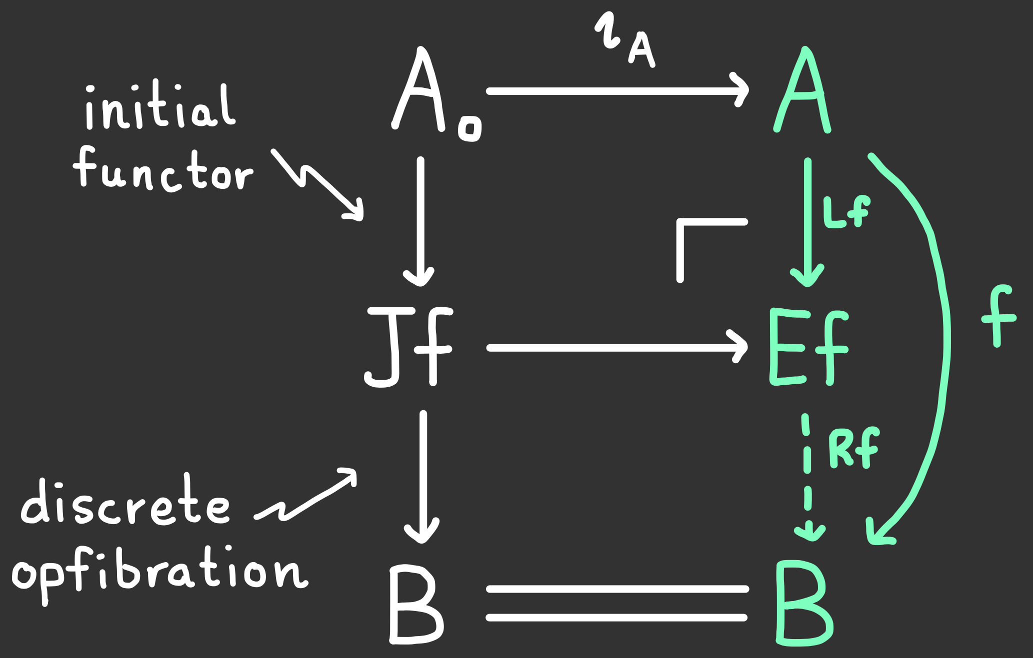

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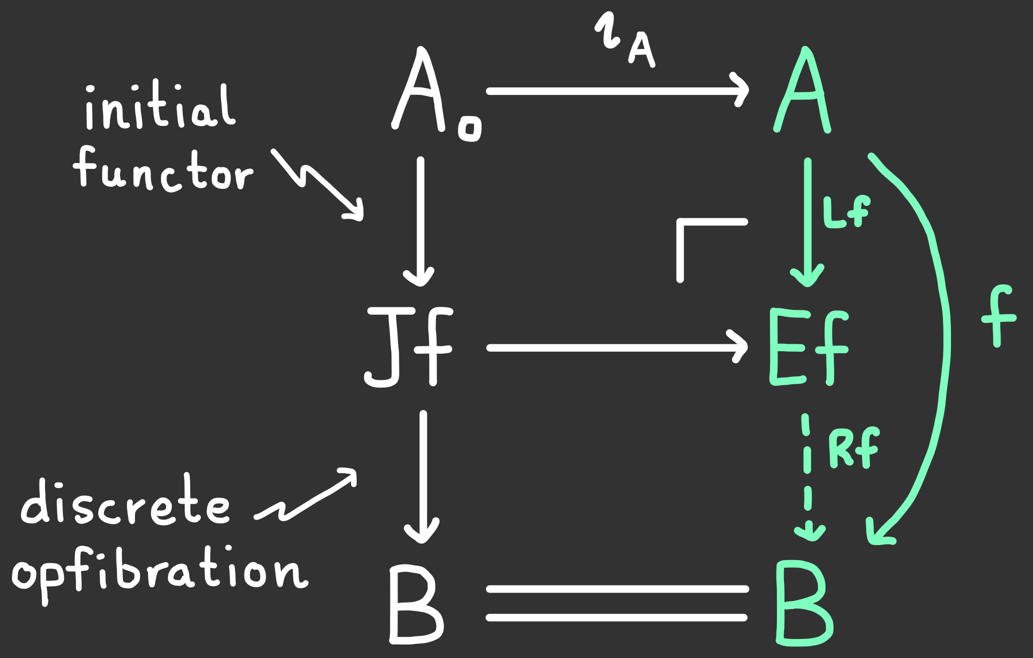

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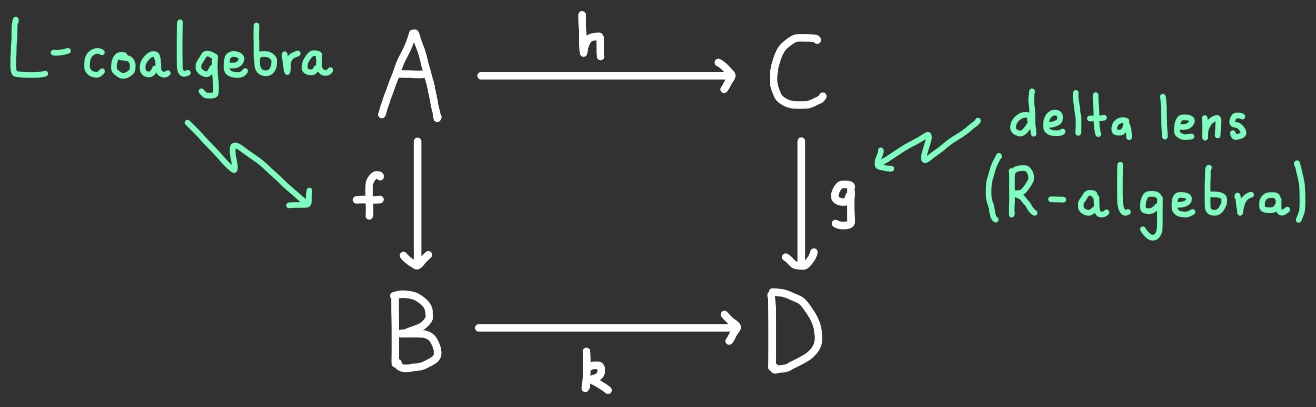

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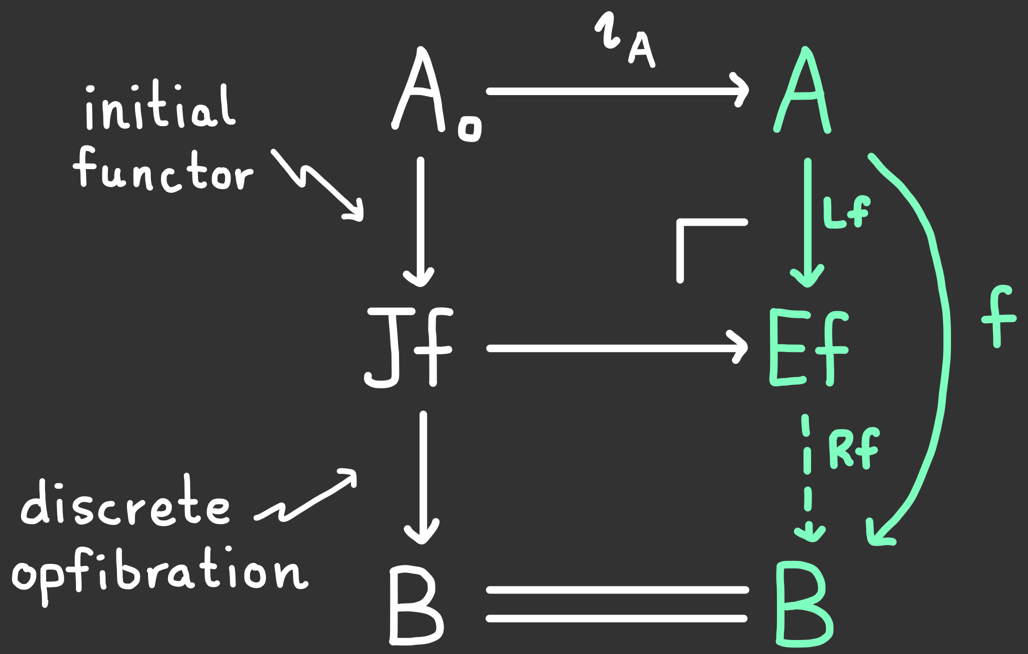

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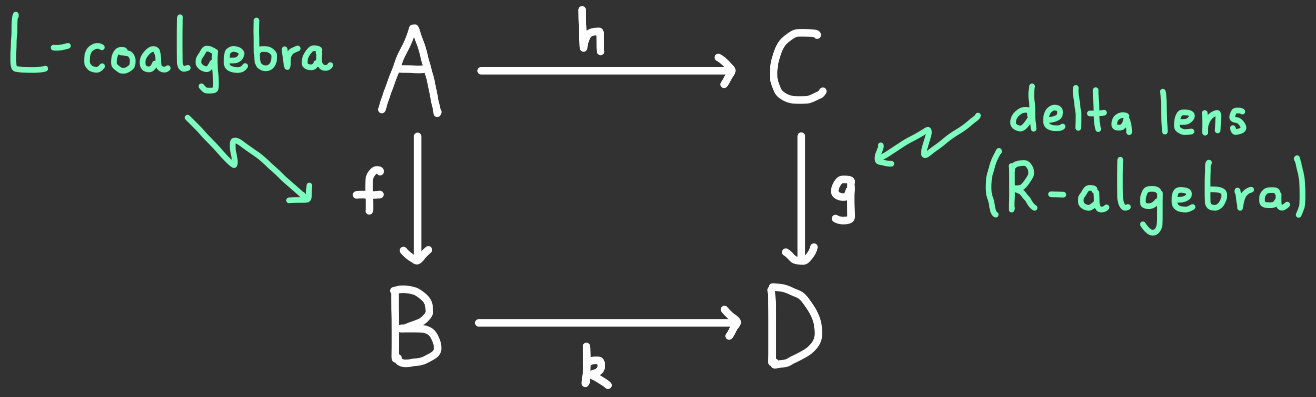

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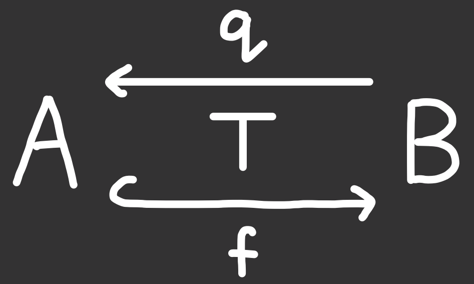
Q: What are the L-coalgebras?

COALGEBRAS ARE "STRUCTURED" LARIs

A L-coalgebra is an adjunction

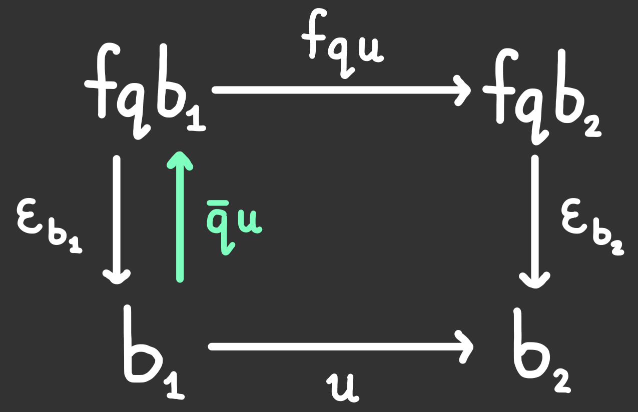


$q \circ f = id_A$

such that if $q(u : b_1 \to b_2) \neq 1$, there is a specified $\bar{q}u : b_1 \to fqb_1$ such that:

$$\bar{q}u \circ \varepsilon_{b_1} = 1$$

$$\varepsilon_{b_2} \circ fqu \circ \bar{q}u = u$$



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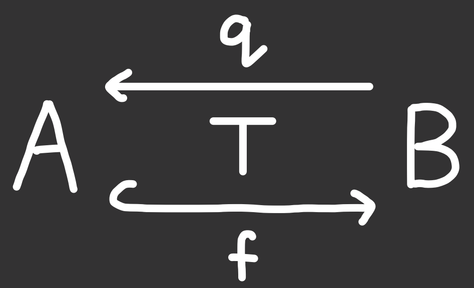

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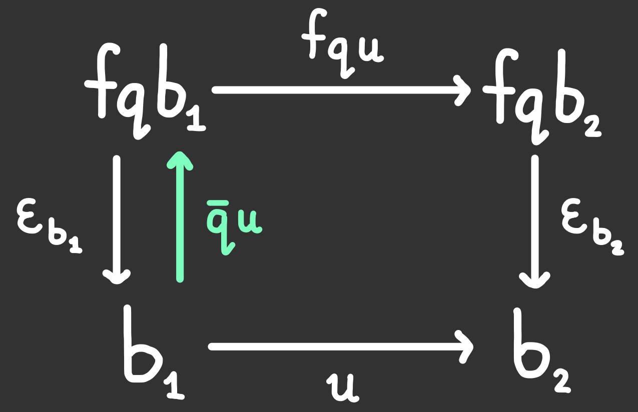
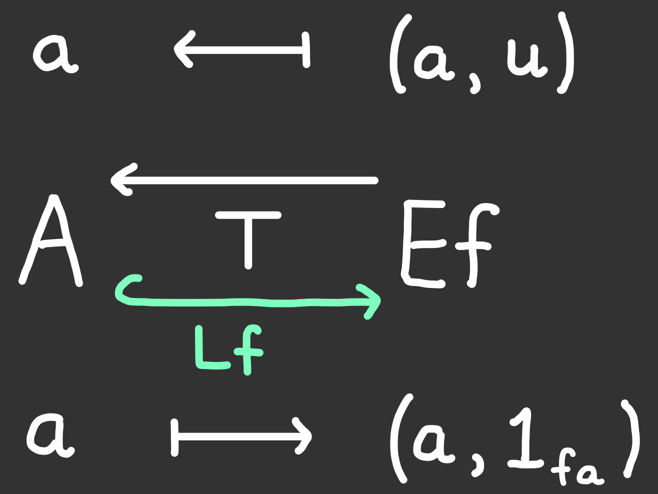
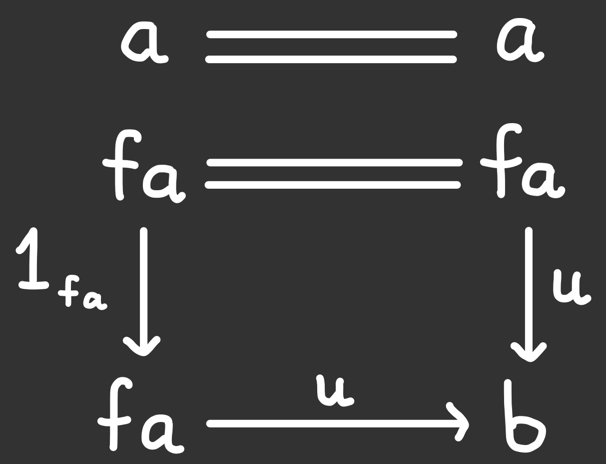


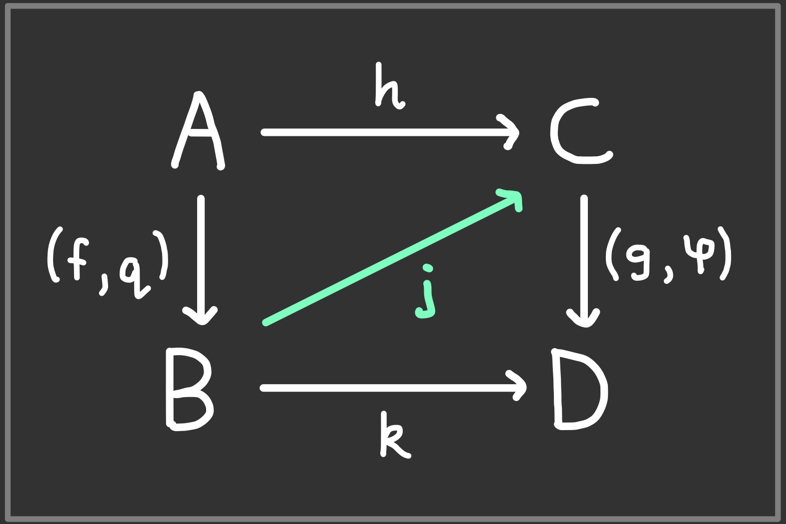
The cofree L-coalgebra on $f : A \to B$ is



with counit:



HOW TO LIFT AGAINST DELTA LENSES (1)



$x \xrightarrow{u} y$

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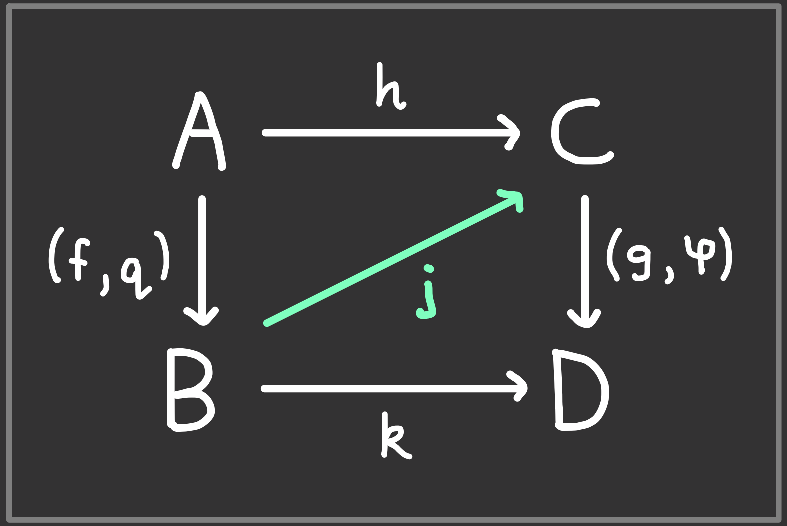

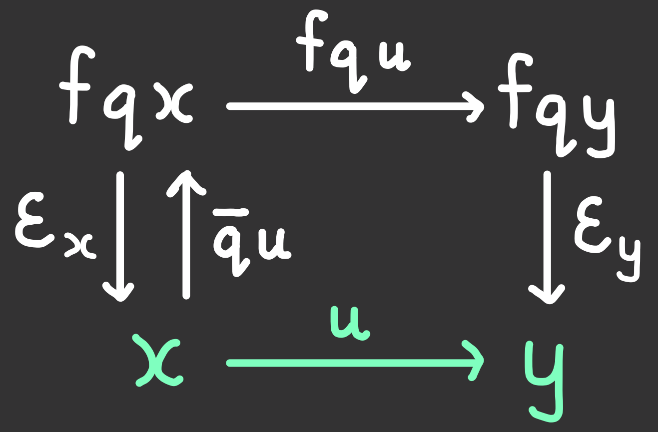

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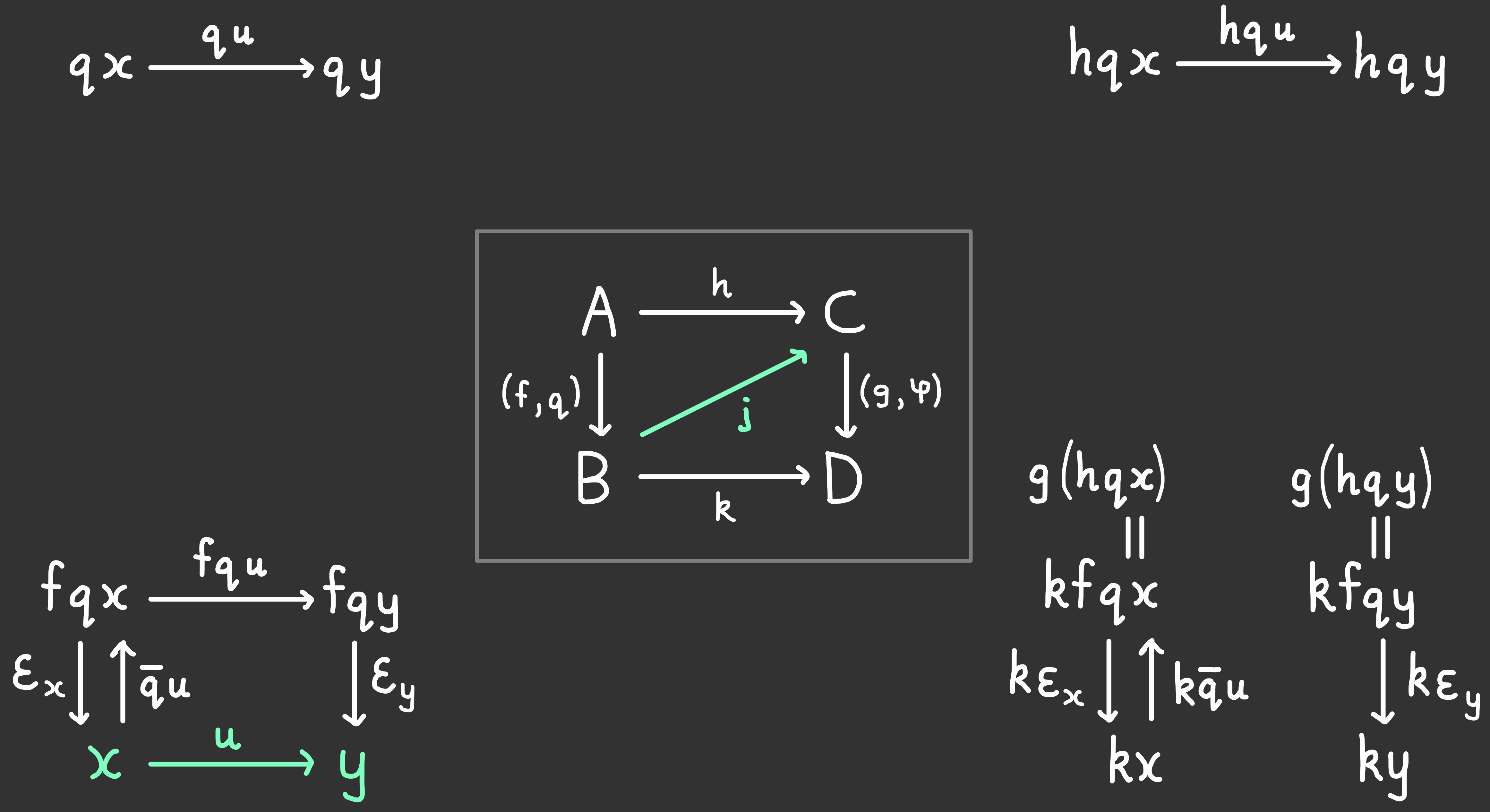

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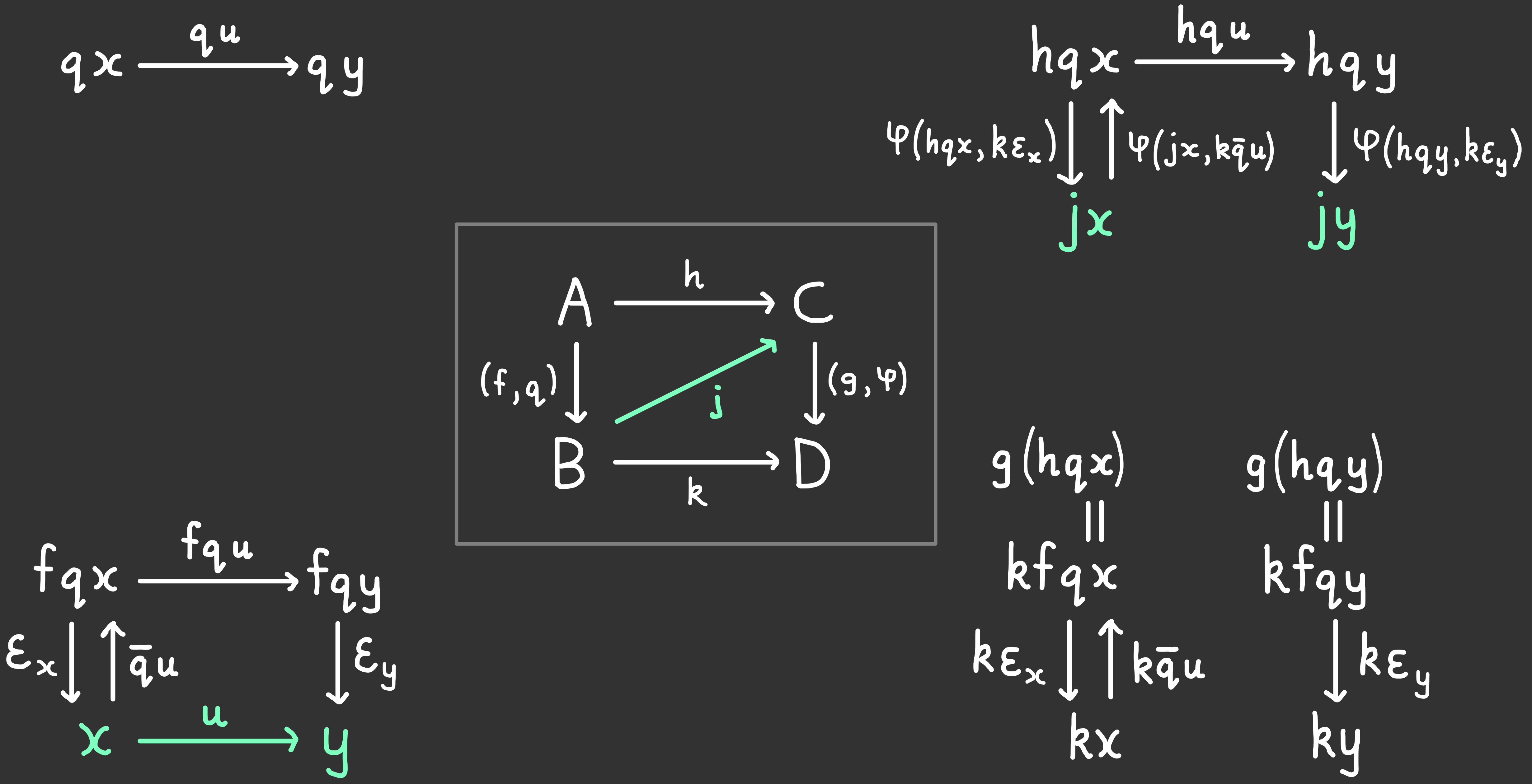

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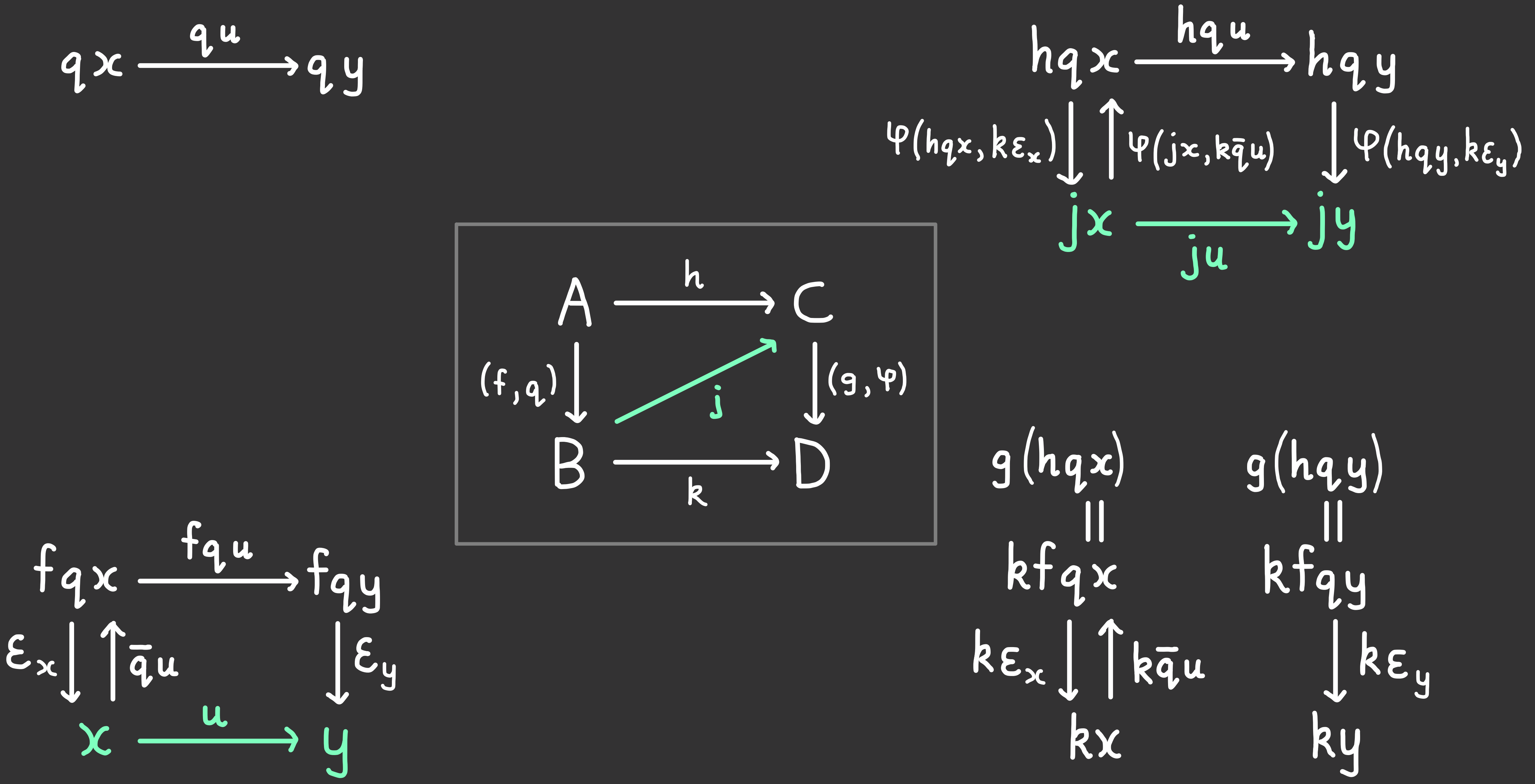

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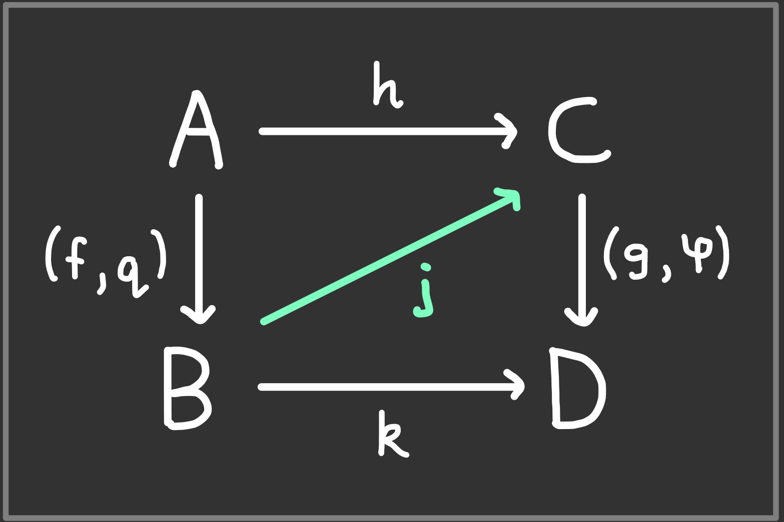

HOW TO LIFT AGAINST DELTA LENSES (2)

$$qx \xrightarrow{\;qu = 1\;} qy$$

$$\begin{array}{ccc} A & \xrightarrow{\;h\;} & C \\ {\scriptstyle (f,q)}\downarrow & \overset{j}{\nearrow} & \downarrow{\scriptstyle (g,\varphi)} \\ B & \xrightarrow[\;k\;]{} & D \end{array}$$

$$\begin{array}{ccc} fqx & = & fqy \\ {\scriptstyle \varepsilon_x}\downarrow & & \downarrow{\scriptstyle \varepsilon_y} \\ x & \xrightarrow{\;u\;} & y \end{array}$$

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$$hqx = hqy$$

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$$\begin{array}{ccc} g(hqx) & & g(hqy) \\ \parallel & & \parallel \\ kfqx & = & kfqy \\ {\scriptstyle k\varepsilon_x}\downarrow & & \downarrow{\scriptstyle k\varepsilon_y} \\ kx & \xrightarrow{\;ku\;} & ky \end{array}$$

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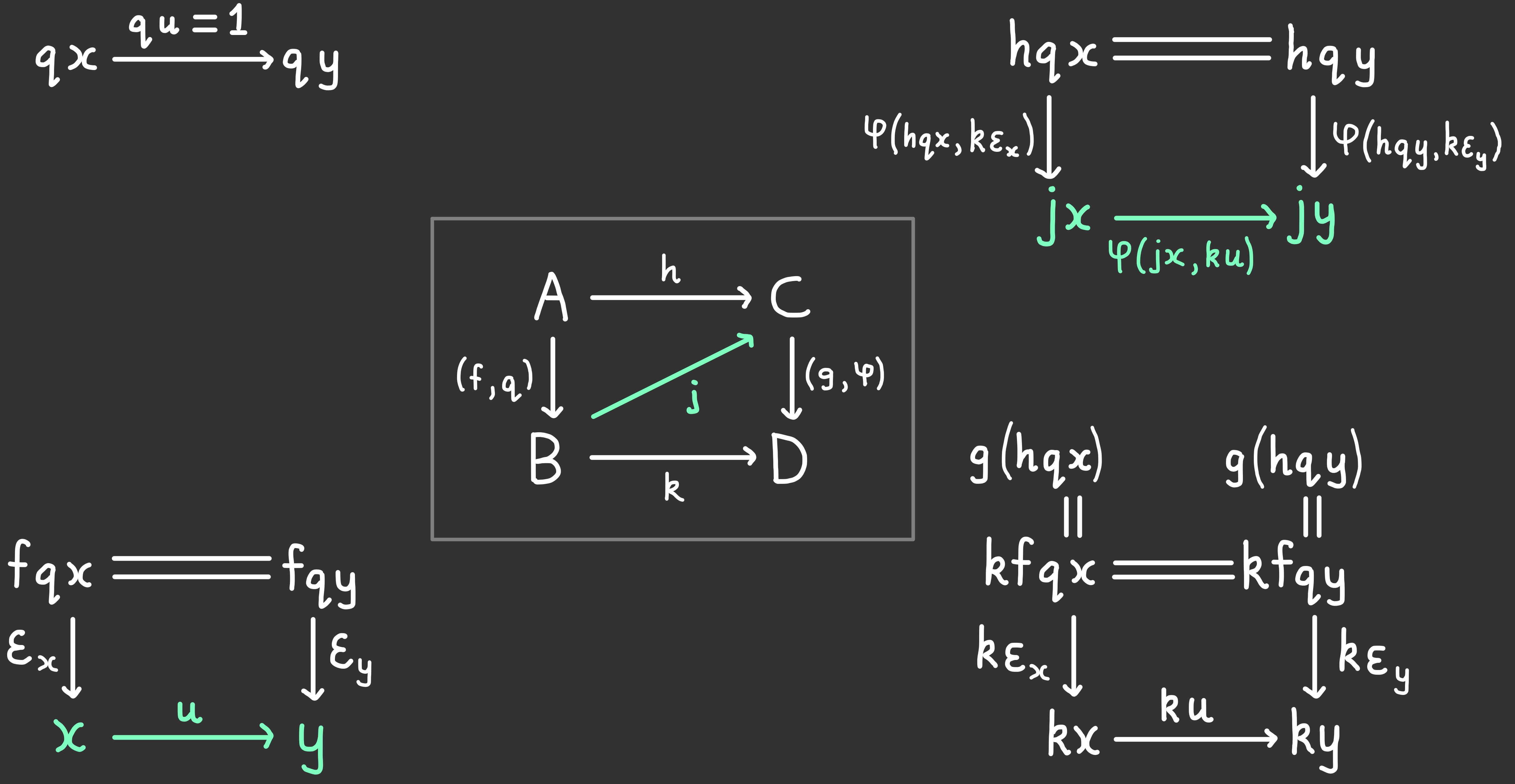

SUMMARY & FURTHER WORK

- We examined two examples of AWFS on Cat whose R-algebras were split opfibrations & delta lenses.
- We constructed explicitly the free delta lens on a functor.
- We characterised the coalgebras that delta lenses lift against as LARIs with extra structure.

SUMMARY & FURTHER WORK

- We examined two examples of AWFS on Cat whose R-algebras were split opfibrations & delta lenses.
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- We characterised the coalgebras that delta lenses lift against as LARIs with extra structure.

- The AWFS for delta lenses generalises to any "nice" category with OFS and idempotent comonad.
- Can assemble a double category of categories, functors, and delta lenses.

Check out the preprint:

arXiv:2305.02732