Coherent differentiation

LHC days 2023

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Linear Logic

Girard’s LL (1986): most (all?) denotational models have an underlying structure similar to *linear algebra*

- tensor product
- linear function spaces
- direct product and coproduct
- duality.

There are also non linear morphisms.
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*Exponential resource modality*: connects the linear and non-linear worlds (categories).

*Dereliction*: we can forget that a function is linear.
Differentiation in LL

Introduces a converse operation.

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  linear $\rightsquigarrow$ non-linear
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  non-linear $\rightsquigarrow$ linear
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reformulating logically the standard laws of the differential calculus.
the differential $\lambda$-calculus:

$$\Gamma \vdash M : A \Rightarrow B \quad \Gamma \vdash N : A$$

$$\Gamma \vdash DM \cdot N : A \Rightarrow B$$

And $DM \cdot N$ is linear in $N$ (and also in $M$).

**Intuition**

The derivative of $M$ should be $M' : A \Rightarrow (A \rightarrow B)$. Then intuitively

$$DM \cdot N = \lambda x : A \cdot (M' x)(N)$$
Strong non-determinism of DiLL

Requires apparently a deduction rule

$$\Gamma \vdash M : A \quad \Gamma \vdash N : A$$

$$\Gamma \vdash M + N : A$$ (++)
Strong non-determinism of DiLL

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\[
\frac{\Gamma \vdash M : A \quad \Gamma \vdash N : A}{\Gamma \vdash M + N : A}
\] (+)

for Leibniz

\[
\frac{df(x, x)}{dx} \cdot u = f'_1(x, x) \cdot u + f'_2(x, x) \cdot u
\]

Interaction between differentiation and contraction.
models of DiLL are additive categories

Leibniz the models $\mathcal{L}$ of DiLL are additive categories:

- $\mathcal{L}(X, Y)$ is a commutative monoid (with additive notations) for each objects $X, Y$ of $\mathcal{L}$
- morphism composition is bilinear.
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- morphism composition is bilinear.

Remark

If $\mathcal{L}$ is cartesian and additive then the cartesian product is also a coproduct, the terminal object is initial: $\& = \oplus$.

some LL degeneracy $=\nonneg$ non-determinism.
... many interesting models of LL are not additive categories.

Remark

One of the main new ideas brought by LL is that the linear/non-linear dichotomy does not require additivity.
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**Question**

Analytic functions have derivatives: what is the status of derivatives in such models?
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In probabilistic coherence spaces ($P_{coh}$) non-linear morphisms are obviously differentiable: they are analytic functions, and $P_{coh}$ is not an additive category.

Question

Analytic functions have derivatives: what is the status of derivatives in such models?

At first sight they seem to live outside...
... a concrete example

In $P_{coh}$ the type 1 of LL is interpreted as

$$[0, 1] \subseteq \mathbb{R}$$
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A non-linear morphism, that is, an element of $\mathcal{L}(1, 1)$, is an analytic function

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In \textbf{Pcoh} the type 1 of LL is interpreted as

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\textbf{Problem}

\[ f'(x) = \sum_{n=0}^{\infty} (n + 1) a_{n+1} x^n \] has no reason to satisfy \( f' \in \mathcal{L}_!(1, 1) \).
For instance: $f$ defined by $f(x) = 1 - \sqrt{1 - x}$ belongs to $\mathbf{P}_{\text{coh}}(1, 1)$
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For instance: $f$ defined by $f(x) = 1 - \sqrt{1-x}$ belongs to $P_{coh}(1,1)$ but

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is not even defined on the whole of $[0, 1]$ and is not bounded on $[0, 1)$.

... and we cannot reject $f$ because it is the interpretation of a program!
Key observation, part 1

If $f \in P_{coh}^!(1,1)$ and $x, u \in [0,1]$ satisfy $x + u \in [0,1]$ then by the Taylor formula at $x$

$$f(x + u) = f(x) + f'(x)u + \frac{1}{2}f''(x)u^2 + \cdots \in [0,1]$$

and all these derivatives are $\geq 0$, so we have $f(x) + f'(x)u \in [0,1]$. 
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If \( f \in P_{\text{coh}}^1(1, 1) \) and

\[ x, u \in [0, 1] \text{ satisfy } x + u \in [0, 1] \]

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and all these derivatives are \( \geq 0 \), so we have

\[
f(x) + f'(x)u \in [0, 1].
\]
So if we set \( S = \{ (x, u) \in [0, 1]^2 \mid x + u \in [0, 1] \} \) we can define

\[
\begin{align*}
Df : S & \rightarrow S \\
(x, u) & \mapsto (f(x), f'(x)u)
\end{align*}
\]

similar to \( T_f \), the tangent bundle functor.
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similar to \( T_f \), the tangent bundle functor.

**Key observation, part 2**

\( S \) can be seen as an object of \( \text{Pcoh} \) and

\[
\forall f \in \text{Pcoh}! (1, 1) \quad Df \in \text{Pcoh}! (S, S)
\]

Can be extended to *all the objects* of \( \text{Pcoh} \), not only 1.

\( \leadsto \) Coherent Differentiation
An important case of coherent differentiation: the elementary situation
Assume:

- $\mathcal{L}$ is a SMCC, tensor $X \otimes Y$, tensor unit 1, internal hom $\mathcal{L}(Z \otimes X, Y) \simeq \mathcal{L}(Z, X \rightarrow Y)$;
Summability in a linear category

Assume:

• \( \mathcal{L} \) is a SMCC, tensor \( X \otimes Y \), tensor unit 1, internal hom \( \mathcal{L}(Z \otimes X, Y) \simeq \mathcal{L}(Z, X \to Y) \); 

• \( \mathcal{L} \) has 0-morphisms \( 0 \in \mathcal{L}(X, Y) \) with \( 0 \cdot t = 0 \) and \( t \cdot 0 = 0 \);
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- \( \mathcal{L} \) is cartesian, cartesian product \( \&_{i \in I} X_i \) with projections \( pr_i \) and if \( (t_i \in \mathcal{L}(Y, X_i))_{i \in I} \) then \( \langle t_i \rangle_{i \in I} \in \mathcal{L}(Y, \&_{i \in I} X_i) \).

We don’t assume \( \mathcal{L} \) to be additive.

Remark (some sums do exist)

\[ \langle \text{Id}_1, 0 \rangle, \langle 0, \text{Id}_1 \rangle \in \mathcal{L}(1 \& 1) \] have a sum

\[ \langle \text{Id}_1, 0 \rangle + \langle 0, \text{Id}_1 \rangle = \langle \text{Id}_1, \text{Id}_1 \rangle \in \mathcal{L}(1 \& 1). \]
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\[ \langle \text{Id}_1, 0 \rangle, \langle 0, \text{Id}_1 \rangle \in \mathcal{L}(1, 1 \& 1) \]
Summability in a linear category

Assume:

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- $\mathcal{L}$ is cartesian, cartesian product $\&_{i \in I} X_i$ with projections $p_r i$ and if $(t_i \in \mathcal{L}(Y, X_i))_{i \in I}$ then $\langle t_i \rangle_{i \in I} \in \mathcal{L}(Y, \&_{i \in I} X_i)$.

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have a sum $\langle \text{Id}_1, 0 \rangle + \langle 0, \text{Id}_1 \rangle = \langle \text{Id}_1, \text{Id}_1 \rangle \in \mathcal{L}(1, 1 \& 1)$. 
The functor of summable pairs

\( S : \mathcal{L} \to \mathcal{L} \) given by

\[
SX = (1 \& 1 \to X)
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**Intuition**

A “point” of \( SX \) is a pair of two points of \( X \) whose sum is well defined.
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A “point” of \( SX \) is a pair of two points of \( X \) whose sum is well defined.

- \( \pi_0 = (\langle \text{Id}_1, 0 \rangle \rightarrow X) \in \mathcal{L}(SX, X) \) fst component of pairs
- \( \pi_1 = (\langle 0, \text{Id}_1 \rangle \rightarrow X) \in \mathcal{L}(SX, X) \) snd component of pairs
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- $\sigma = (\langle \text{Id}_1, \text{Id}_1 \rangle \rightarrow X) \in \mathcal{L}(SX, X)$ sum of pairs.
Assume \( \langle \text{Id}_1, 0 \rangle, \langle 0, \text{Id}_1 \rangle \in \mathcal{L}(1, 1 \& 1) \) are jointly epic and hence \( \pi_0, \pi_1 \in \mathcal{L}(S X, X) \) are jointly monic: this is a property of \( \mathcal{L} \) which holds very often.
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**Definition (summability and sum of morphisms)**

$f_0, f_1 \in \mathcal{L}(Y, X)$ are summable if there is $h \in \mathcal{L}(Y, S\!X)$ such that $\pi_i \circ h = f_i$ ($i = 0, 1$).
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This $h$ is unique: $\langle f_0, f_1 \rangle_S = h$ (witness of summability).
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\( f_0 + f_1 = \sigma \langle f_0, f_1 \rangle_s \).
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**Fact**

*If \( \mathcal{L} \) satisfies an additional witness property then, equipped with 0 and +, each \( \mathcal{L}(X, Y) \) is a commutative partial monoid.*
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**Definition (summability and sum of morphisms)**

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**Fact**

If \( \mathcal{L} \) satisfies an additional witness property then, equipped with 0 and +, each \( \mathcal{L}(X, Y) \) is a commutative partial monoid.

Composition is compatible with this structure.
Comonoid structure of $1 \& 1$

When $\mathcal{L}$ satisfies these conditions, $1 \& 1$ has a structure of commutative comonoid

$$\text{pr}_0 : 1 \& 1 \rightarrow 1 \quad \text{fst projection of } \&$$

$$\widetilde{\mathcal{L}} : 1 \& 1 \rightarrow (1 \& 1) \otimes (1 \& 1)$$

fully characterized by

$$\widetilde{\mathcal{L}} \langle \text{Id}_1, 0 \rangle = \langle \text{Id}_1, 0 \rangle \otimes \langle \text{Id}_1, 0 \rangle$$

$$\widetilde{\mathcal{L}} \langle 0, \text{Id}_1 \rangle = \langle \text{Id}_1, 0 \rangle \otimes \langle 0, \text{Id}_1 \rangle + \langle 0, \text{Id}_1 \rangle \otimes \langle \text{Id}_1, 0 \rangle$$

NB: this sum is well defined (by the witness assumption).

Remember that $\langle \text{Id}_1, 0 \rangle$ and $\langle 0, \text{Id}_1 \rangle$ are jointly epic.
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NB: this sum is well defined (by the witness assumption).

Remember that $\langle \text{id}_1, 0 \rangle$ and $\langle 0, \text{id}_1 \rangle$ are jointly epic.
Assume that $\mathcal{L}$ is equipped with a **resource modality**, that is

- a comonad $(\_!, \text{der}, \text{dig})$
- with a symmetric monoidal structure from $(\mathcal{L}, \&)$ to $(\mathcal{L}, \otimes)$: there are well-behaved isos $1 \to \top \_!$ and $\_! X \otimes \_! Y \to \_!(X \& Y)$. 

Exponential
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  - there are well-behaved isos $1 \to !\top$ and $!X \otimes !Y \to !(X \& Y)$.

Then the Kleisli category $\mathcal{L}_!$ is intuitively the category of non-linear morphisms that we will differentiate.

- $\text{Obj}(\mathcal{L}_!) = \text{Obj}(\mathcal{L})$
- $\mathcal{L}_!(X, Y) = \mathcal{L}(!X, Y)$. 

Exponential
Definition

A differential structure on $\mathcal{L}$ is a $!$-coalgebra structure $\tilde{\partial}$ on $1$ & $1$:

$$\tilde{\partial} : 1 \& 1 \to !(1 \& 1)$$

such that $\text{pr}_0$ and $\tilde{L}$ are coalgebra morphisms.
Differential structure

**Definition**

A **differential structure** on \( \mathcal{L} \) is a \(!\)-coalgebra structure \( \tilde{\partial} \) on \( 1 \& 1 \):

\[
\tilde{\partial} : 1 \& 1 \rightarrow !((1 \& 1))
\]

such that \( \text{pr}_0 \) and \( \tilde{L} \) are coalgebra morphisms.

**Remark (CD is everywhere...)**

If \((\mathcal{L}, !\_)_\) is a Lafont category (ie. \(!\_\) is the cofree symmetric comonoid functor) there is exactly one differential structure, induced by \((\text{pr}_0, \tilde{L})\).
What is the link with differentiation?

Using \( \tilde{\partial} \) we can define a natural transformation
\[
\partial_X : !S_X = !(1 \& 1 \rhd X) \to S!X = (1 \& 1 \rhd !X),
\]
What is the link with differentiation?

Using \( \partial \) we can define a natural transformation

\[
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\]

Curry transpose of

\[
!(1 \& 1 \twoheadrightarrow X) \otimes (1 \& 1) \xrightarrow{\text{Id} \otimes \partial} !(1 \& 1 \twoheadrightarrow X) \otimes !(1 \& 1) \xrightarrow{\mu^2} !((1 \& 1 \twoheadrightarrow X) \otimes (1 \& 1)) \xrightarrow{!ev} !X
\]

\( \mu^2 \): lax monoidality \( \otimes \rightarrow \otimes \), derived from the monoidality \( \& \rightarrow \otimes \).

\( \text{ev} \): evaluation morphism.
Fact (extending $S$ to $L_!$ thanks to $\partial \rightsquigarrow$ differentiation functor)

$\partial_X : !SX \to S!X$ is a distributive law.
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If $t \in \mathcal{L}_!(X, Y) = \mathcal{L}(!X, Y)$ seen as a non-linear morphism $X \rightarrow Y$ then

$$D t = (S t) \partial_X \in \mathcal{L}_!(SX, SY)$$
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can be understood intuitively as mapping $(x, u) \in S X$ (that is $x, u \in X$ summable) to $(t(x), t'(x) \cdot u) \in S Y$, a summable pair.
Fact (extending $S$ to $\mathcal{L}_!$ thanks to $\partial \mapsto$ differentiation functor)

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If $t \in \mathcal{L}_!(X, Y) = \mathcal{L}(!X, Y)$ seen as a non-linear morphism $X \to Y$ then

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can be understood intuitively as mapping $(x, u) \in SX$ (that is $x, u \in X$ summable) to $(t(x), t'(x) \cdot u) \in SY$, a summable pair.

$D$ is a functor (chain rule).
The simplest example: strict coherence spaces

\[ E = (|E|, \bowtie_E) \] where \(|E|\) is a set (web) and \(\bowtie_E\) is a binary and symmetric relation on \(|E|\) (not required to be reflexive nor anti-reflexive).

\[ \text{Cl}(E) = \{ x \subseteq |E| \mid \forall a, a' \in x \quad a \bowtie_E a' \} \]

\[ E \mapsto F \text{ defined by } |E \mapsto F| = |E| \times |F| \text{ and } (a, b) \bowtie_{E \mapsto F} (a', b') \text{ if } a \bowtie_E b \Rightarrow a' \bowtie_F b' \].

Category \( \text{Scoh} \): objects are the strict coherence spaces and \( \text{Scoh}(E, F) = \text{Cl}(E \mapsto F) \subseteq |E| \times |F| \).


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The SMC structure of SCS

- $|1| = \{\ast\}$ with $\ast \bowtie_1 \ast$
- $|E \otimes F| = |E| \times |F|$ and $(a, b) \bowtie_{E \otimes F} (a', b')$ if $a \bowtie_E a'$ and $b \bowtie_F b'$. 
The SMC structure of SCS

- $|1| = \{\ast\}$ with $\ast \sqcup_1 \ast$
- $|E \otimes F| = |E| \times |F|$ and $(a, b) \sqcup_{E \otimes F} (a', b')$ if $a \sqcup_E a'$ and $b \sqcup_F b'$.
- SMCC: $\mathbf{Scoh}(G \otimes E, F) \sim \mathbf{Scoh}(G, E \rightarrow F)$ trivially maps $t$ to $\{(c, (a, b)) \mid ((c, a), b) \in t\}$. 
Cartesian product

- $|\&_{i \in I} E_i| = \bigcup_{i \in I} \{i\} \times |E_i|$
- $(i, a) \&_{i \in I} E_i (j, b) \text{ if } i = j \Rightarrow a \&_{E_i} b.$
Cartesian product

- $|\&_{i \in I} E_i| = \bigcup_{i \in I} \{i\} \times |E_i|$
- $(i, a) \cap_{\&_{i \in I} E_i} (j, b)$ if $i = j \Rightarrow a \cap_{E_i} b$.
- So that in particular $\text{Cl}(\&_{i \in I} E_i) \cong \prod_{i \in I} \text{Cl}(E_i)$.

**Fact**

$|1 \& 1| = \{0, 1\}$ with $i \cap_{1\&1} j$ for all $i, j \in \{0, 1\}$, so that $\text{Cl}(1 \& 1) = \mathcal{P}(\{0, 1\})$. 

$\langle \text{Id}_1, 0 \rangle = \{\langle *, 0\rangle\}$ and $\langle 0, \text{Id}_1 \rangle = \{\langle *, 1\rangle\}$ are trivially jointly epic.
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$s_0, s_1 \in \text{Scoh}(E, F)$ are summable iff $s_0 \cup s_1 \in \text{Scoh}(E, F)$ and then $s_0 + s_1 = s_0 \cup s_1$. 
Intermezzo: duality and booleans

**Scoh** is a model of classical LL: take $|E^\bot| = |E|$ and $a \sim_{E^\bot} b$ if $\neg(a \sim_E b)$. Then $E^{\bot\bot} = E$. 

**Remark (SCS are not a stable model)** Contrarily to Girard’s CS, SCS accept the parallel or program.
Intermezzo: duality and booleans

\textbf{ScOH} is a model of classical LL: take $|E\perp| = |E|$ and $a \bowtie_{E\perp} b$ if $\neg(a \bowtie_E b)$. Then $E\perp\perp = E$.

So \textbf{ScOH} has coproducts, in particular $1 \oplus 1 = (1\perp \& 1\perp)\perp$.
**Intermezzo: duality and booleans**

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$\text{Cl}(1 \oplus 1) = \{\emptyset, \{0\}, \{1\}\}$ so $\{0\}$ and $\{1\}$ are not summable in $1 \oplus 1$ (though they are summable in $1 \& 1$): the category **Scoh** is not additive.
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Contrarily to Girard’s CS, SCS accept the *parallel or* program.
Comonoid structure of 1 & 1

Remember

• $|1| = \{\ast\}$ and $\ast \bowtie_1 \ast$
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1 & 1 as a comonoid

- counit: $\text{pr}_0 = \{(0, \ast) \in \text{Scoh}(1 \& 1, 1)\}$
- comultiplication: $\tilde{L} \in \text{Scoh}(1 \& 1, (1 \& 1) \otimes (1 \& 1))$ given by
  $\tilde{L} = \{(0, (0, 0))\} \cup \{(1, (0, 1)), (1, (1, 0))\}$
The cofree exponential: **Scoh** is Lafont

Much simpler than the exponential of Lamarche who insisted on $|!E| \subseteq P_{\text{fin}}(|E|)$. 

Then $e_\partial \in \text{Scoh}(1 \& 1, !1 \& 1)$ is $e_\partial = \{ (i, [i_1, \ldots, i_k]) \mid i, i_1, \ldots, i_k \in \{0, 1\} \text{ and } i = i_1 + \cdots + i_k \}$. 

- either $i = 0$ and all the $i_j$'s are $0$
- or $i = 1$ and all the $i_j$'s $= 0$ but one which $= 1$. 


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Instead we use finite multisets: $|!E| = \mathcal{M}_{\text{fin}}(|E|)$ and

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Induced differentiation

Remember that $SE = (1 \& 1 \rightarrow E)$.

So that $|SE| = \{0, 1\} \times |E|$ and $(i, a) \triangleright_{SE} (j, b) \iff a \triangleright_{E} b.$
Induced differentiation

Remember that \( SE = (1 \& 1 \to E) \).

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Given \( t \in \text{Scoh}(!E, F) \) we get \(Dt \in \text{Scoh}(!SE, SF)\).
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Given $t \in \text{Scoh}(!E, F)$ we get $Dt \in \text{Scoh}(!SE, SF)$. Remember that intuitively

$$Dt(x, u) = (t(x), t'(x) \cdot u).$$
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**Fact**

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Dt = \{([0, a_1, \ldots, (0, a_n)], (0, b)) \mid ([a_1, \ldots, a_n], b) \in t\}
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Remember that \( SE = (1 \& 1 \leadsto E) \).

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Fact

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Dt = \{((0, a_1, \ldots, (0, a_n)), (0, b)) \mid ([a_1, \ldots, a_n], b) \in t\}
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\[
\cup \{(((0, a_1, \ldots, (0, a_n), (1, a)), (1, b)) \mid ([a_1, \ldots, a_n, a], b) \in t\}
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• We are developing the general theory of coherent differentiation with Aymeric Walch (PhD thesis).
Conclusion

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- There is also a syntactic version of CD, a “CD PCF” which
  - has a differentiation operation at all types
  - as well as general recursion (fixpoint operators at all types)
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Conclusion

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  - has a differentiation operation at all types
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- It was impossible to have all these features in the differential \( \lambda \)-calculus.