Coherent differentiation

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Linear Logic

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- tensor product
- linear function spaces
- direct product and coproduct
- duality.

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Dereliction: we can forget that a function is linear.

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reformulating logically the standard laws of the differential calculus.

$\rightsquigarrow$ the differential λ-calculus:

$$\frac{\Gamma \vdash M : A \Rightarrow B \qquad \Gamma \vdash N : A}{\Gamma \vdash \mathrm{D}M \cdot N : A \Rightarrow B}$$

And $\mathrm{D}M \cdot N$ is linear in N (and also in M).

Intuition

The derivative of M should be $M' : A \Rightarrow (A \multimap B)$. Then intuitively

$$\mathrm{D}M \cdot N = \lambda x : A \cdot (M'\, x)(N)$$

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for Leibniz

$$\frac{df(x,x)}{dx} \cdot u = f_1'(x,x) \cdot u + f_2'(x,x) \cdot u$$

Interaction between differentiation and contraction.

$\rightsquigarrow$ models of DiLL are additive categories

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Remark

If $\mathcal{L}$ is cartesian and additive then the cartesian product is also a coproduct, the terminal object is initial: $\& = \oplus$.

$\rightsquigarrow$ some LL degeneracy = non-determinism.

But. . .

. . . many interesting models of LL *are not* additive categories.

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At first sight they seem to live outside. . .

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In **Pcoh** the type 1 of LL is interpreted as

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Problem

$f'(x) = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$ has no reason to satisfy $f' \in \mathcal{L}_!(1,1)$.

For instance: f defined by $f(x) = 1 - \sqrt{1 - x}$ belongs to $\mathbf{Pcoh}_!(1, 1)$

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... and we cannot reject f because it is the interpretation of a program!

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and all these derivatives are ≥ 0, so we have

$$f(x) + f'(x)u \in [0,1] \,.$$

So if we set $S = \{(x, u) \in [0, 1]^2 \mid x + u \in [0, 1]\}$ we can define

$$\begin{aligned} \mathrm{D}f : S &\to S \\ (x, u) &\mapsto (f(x), f'(x)u) \end{aligned}$$

similar to $\mathrm{T}f$, the tangent bundle functor.

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Key observation, part 2

S can be seen as an object of **Pcoh** and

$$\forall f \in \mathbf{Pcoh}_!(1,1) \quad \mathrm{D}f \in \mathbf{Pcoh}_!(S,S)$$

Can be extended to *all the objects* of **Pcoh**, not only 1.

$\rightsquigarrow$ Coherent Differentiation

An important case of coherent differentiation: the elementary situation

Summability in a linear category

Assume:

- $\mathcal{L}$ is a SMCC, tensor $X \otimes Y$, tensor unit 1, internal hom $\mathcal{L}(Z \otimes X, Y) \simeq \mathcal{L}(Z, X \multimap Y)$;

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$\langle \mathrm{Id}_1, 0 \rangle, \langle 0, \mathrm{Id}_1 \rangle \in \mathcal{L}(1, 1 \,\&\, 1)$
have a sum $\langle \mathrm{Id}_1, 0 \rangle + \langle 0, \mathrm{Id}_1 \rangle = \langle \mathrm{Id}_1, \mathrm{Id}_1 \rangle \in \mathcal{L}(1, 1 \,\&\, 1)$.

The functor of summable pairs

$\mathbf{S} : \mathcal{L} \to \mathcal{L}$ given by

$$\mathbf{S}X = (1 \mathbin{\&} 1 \multimap X)$$

Intuition

A "point" of $\mathbf{S}X$ is a pair of two points of X whose sum is well defined.

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- $\sigma = (\langle \mathsf{Id}_1, \mathsf{Id}_1 \rangle \multimap X) \in \mathcal{L}(\mathbf{S}X, X)$ sum of pairs.

Assume $\langle \mathsf{Id}_1, 0\rangle, \langle 0, \mathsf{Id}_1\rangle \in \mathcal{L}(1, 1 \mathbin{\&} 1)$ are jointly epic and hence $\pi_0, \pi_1 \in \mathcal{L}(\mathbf{S}X, X)$ are jointly monic: this is a property of $\mathcal{L}$ which holds very often.

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$f_0, f_1 \in \mathcal{L}(Y, X)$ are summable if there is $h \in \mathcal{L}(Y, \mathbf{S}X)$ such that $\pi_i\ h = f_i\ (i = 0, 1)$.

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Composition is compatible with this structure.

Comonoid structure of $1 \& 1$

When $\mathcal{L}$ satisfies these conditions, $1 \& 1$ has a structure of commutative comonoid

$$\mathsf{pr}_0 : 1 \& 1 \to 1 \quad \text{fst projection of } \&$$
$$\widetilde{\mathsf{L}} : 1 \& 1 \to (1 \& 1) \otimes (1 \& 1)$$

fully characterized by

$$\widetilde{\mathsf{L}} \langle \mathsf{Id}_1, 0 \rangle = \langle \mathsf{Id}_1, 0 \rangle \otimes \langle \mathsf{Id}_1, 0 \rangle$$
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NB: this sum is well defined (by the witness assumption).

Remember that $\langle \mathsf{Id}_1, 0\rangle$ and $\langle 0, \mathsf{Id}_1\rangle$ are jointly epic.

Exponential

Assume that $\mathcal{L}$ is equipped with a resource modality, that is

- a comonad $(!_{\_}, \mathrm{der}, \mathrm{dig})$
- with a symmetric monoidal structure from $(\mathcal{L}, \&)$ to $(\mathcal{L}, \otimes)$: there are well-behaved isos $1 \to !\top$ and $!X \otimes !Y \to !(X \& Y)$.

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Then the Kleisli category $\mathcal{L}_!$ is intuitively the category of non-linear morphisms that we will differentiate.

- $\mathrm{Obj}(\mathcal{L}_!) = \mathrm{Obj}(\mathcal{L})$
- $\mathcal{L}_!(X, Y) = \mathcal{L}(!X, Y)$.

Differential structure

Definition

A differential structure on $\mathcal{L}$ is a !-coalgebra structure $\widetilde{\partial}$ on $1 \,\&\, 1$:

$$\widetilde{\partial} : 1 \,\&\, 1 \rightarrow !(1 \,\&\, 1)$$

such that pr_0 and $\widetilde{\mathsf{L}}$ are coalgebra morphisms.

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Remark (CD is everywhere. . .)

If $(\mathcal{L}, !_{\_})$ is a Lafont category (ie. $!_{\_}$ is the cofree symmetric comonoid functor) there is exactly one differential structure, induced by $(\mathrm{pr}_0, \widetilde{\mathrm{L}})$.

What is the link with differentiation?

Using $\widetilde{\partial}$ we can define a natural transformation $\partial_X : !\mathbf{S}X = !(1 \mathbin{\&} 1 \multimap X) \to \mathbf{S}!X = (1 \mathbin{\&} 1 \multimap !X)$,

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Curry transpose of

$$\begin{array}{c} !(1 \mathbin{\&} 1 \multimap X) \otimes (1 \mathbin{\&} 1) \\ \big\downarrow \mathrm{Id} \otimes \widetilde{\partial} \\ !(1 \mathbin{\&} 1 \multimap X) \otimes !(1 \mathbin{\&} 1) \\ \big\downarrow \mu^2 \\ !((1 \mathbin{\&} 1 \multimap X) \otimes (1 \mathbin{\&} 1)) \\ \big\downarrow !\mathrm{ev} \\ !X \end{array}$$

μ^2: lax monoidality $\otimes \to \otimes$, derived from the monoidality $\& \to \otimes$.
ev: evaluation morphism.

Fact (extending $\mathbf{S}$ to $\mathcal{L}_!$ thanks to $\partial \rightsquigarrow$ differentiation functor)

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can be understood intuitively as mapping $(x, u) \in \mathbf{S}X$ *(that is* $x, u \in X$ *summable) to* $(t(x), t'(x) \cdot u) \in \mathbf{S}Y$*, a summable pair.*

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$\mathbf{D}$ *is a functor (chain rule).*

The simplest example: strict coherence spaces

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$E = (|E|, \frown_E)$ where $|E|$ is a set (web) and $\frown_E$ is a binary and symmetric relation on $|E|$ (not required to be reflexive nor anti-reflexive).

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Composition: relational composition. Identity: diagonal relation.

The SMC structure of SCS

- $|1| = \{*\}$ with $* \frown_1 *$
- $|E \otimes F| = |E| \times |F|$ and $(a, b) \frown_{E \otimes F} (a', b')$ if $a \frown_E a'$ and $b \frown_F b'$.

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- SMCC: $\mathbf{Scoh}(G \otimes E, F) \simeq \mathbf{Scoh}(G, E \multimap F)$ trivially maps t to $\{(c, (a, b)) \mid ((c, a), b) \in t\}$.

Cartesian product

- $|\&_{i \in I} E_i| = \bigcup_{i \in I} \{i\} \times |E_i|$
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$s_0, s_1 \in \mathbf{Scoh}(E,F)$ *are summable iff* $s_0 \cup s_1 \in \mathbf{Scoh}(E,F)$ *and then* $s_0 + s_1 = s_0 \cup s_1$.

Intermezzo: duality and booleans

Scoh is a model of classical LL: take $|E^\perp| = |E|$ and $a \frown_{E^\perp} b$ if $\neg(a \frown_E b)$. Then $E^{\perp\perp} = E$.

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Remark (SCS are not a stable model)

Contrarily to Girard's CS, SCS accept the *parallel or* program.

Comonoid structure of $1 \& 1$

Remember

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$1 \mathbin{\&} 1$ as a comonoid

- counit: $\mathrm{pr}_0 = \{(0, *) \in \mathbf{Scoh}(1 \mathbin{\&} 1, 1)\}$
- comultiplication: $\widetilde{\mathsf{L}} \in \mathbf{Scoh}(1 \mathbin{\&} 1, (1 \mathbin{\&} 1) \otimes (1 \mathbin{\&} 1))$ given by

$$\widetilde{\mathsf{L}} = \{(0, (0, 0))\} \cup \{(1, (0, 1)), (1, (1, 0))\}$$

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Then $\widetilde{\partial} \in \mathbf{Scoh}(1 \mathbin{\&} 1, !(1 \mathbin{\&} 1))$ is

$$\widetilde{\partial} = \{(i, [i_1, \ldots, i_k]) \mid i, i_1, \ldots, i_k \in \{0, 1\} \text{ and } i = i_1 + \cdots + i_k\}$$

that is

- either $i = 0$ and all the i_j's are $= 0$
- or $i = 1$ and all the i_j's $= 0$ but one which $= 1$.

Induced differentiation

Remember that $\mathbf{S}E = (1 \mathbin{\&} 1 \multimap E)$.

So that $|\mathbf{S}E| = \{0, 1\} \times |E|$ and $(i, a) \frown_{\mathbf{S}E} (j, b) \Leftrightarrow a \frown_E b$.

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Fact

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  It was impossible to have all these features in the differential λ-calculus.