# Polygraphic homology of categories with local coefficients

Léonard Guetta
(Joint work with Georges Maltsiniotis)

Max Planck Institute - Bonn

Journées François Métayer
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# Introduction

## In the previous episode

Let $X$ be an $\omega$-category and $M\colon X^{\mathrm{op}} \to \mathrm{Ab}$ a local coefficient system on $X$. We have a canonical morphism

$$\underbrace{H_*(N(X), M)}_{\text{Homology of the nerve}} \longrightarrow \underbrace{H_*^{\mathrm{pol}}(X, M)}_{\text{Polygraphic homology}} \qquad (1)$$

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By analogy, (1) should be an isomorphism. But it is *not* in general.

# The bubble

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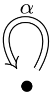
$\mathbb{B} :=$ 

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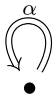
$\mathbb{B} :=$ 


We have

$$H_k^{\mathrm{pol}}(\mathbb{B}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

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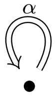
$\mathbb{B} :=$ 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But $N(\mathbb{B})$ has the homotopy type of a $K(\mathbb{Z}, 2)$, so

$$H_k(N(\mathbb{B}), \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

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Then, in the "resolution", there should be a 4-dimensional cell

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Then, in the "resolution", there should be a 4-dimensional cell

$$\mathrm{id}_{\alpha*\alpha} \Rightarrow_4 (\mathrm{EH}).$$

This cell generates non-trivial homology in dimension 4.

# Goal of this talk

**Theorem (Maltsiniotis & G., 2023)**

Let $X$ be a 1-**category** and $M\colon X^{\mathrm{op}} \to \mathrm{Ab}$ a local coefficient system on $X$, then

$$H_*(N(X), M) \longrightarrow H_*^{\mathrm{pol}}(X, M)$$

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For *constant* coefficient and $X$ *monoid*, this was proven by Lafont and Métayer (2009).

**Remark:** The (polygraphic) homology of a 1-category need *not* be trivial above dimension 1. This is because we have to take a polygraphic resolution

$$P \to X,$$

but $P$ is generally not a 1-category.

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# Yet another definition of homology

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## Definition

Let $X$ be a 1-category and $M\colon X^{\mathrm{op}} \to \mathrm{Ab}$ a local coefficient system on $X$. We define the (total) homology of $X$ with coefficient in $M$ as the following object of $\mathrm{Der}(Ab)$

$$H_*(X, M) := \operatorname*{hocolim}_{x \in X^{\mathrm{op}}} M_x,$$

where $M$ is seen as a functor $X^{\mathrm{op}} \to \mathrm{Comp}(\mathrm{Ab})$ concentrated in degree 0 and $\mathrm{hocolim}$ is the left derived functor of the colimit.

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Slogan : The homology is the derived colimit.

# Example:

Let $G$ be a group and $M\colon G^{\mathrm{op}} \to \mathsf{Ab}$ a right $G$-module. By definition

$$H_n(G, M) = \mathrm{Tor}_n^{\mathbb{Z}[G]}(M, \mathbb{Z}),$$

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Hence, we recover

$$H_*(G, M) = \mathbb{L}\underset{G^{\mathrm{op}}}{\mathrm{colim}}\, M =: \underset{G^{\mathrm{op}}}{\mathrm{hocolim}}\, M$$

# Strategy

The strategy is to use third definition of homology to connect the homology of the nerve with the polygraphic homology

$$H_*(N(X), M) \simeq \operatorname*{hocolim}_{x \in X^{\mathrm{op}}} M_x \simeq H_*^{\mathrm{pol}}(X, M).$$

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# Homology of the nerve as homotopy colimit

**Proposition**

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**Sketch of proof:**

- Observation 1: $(X, 1_X)$ is the colimit of

$$X^{\mathrm{op}} \to \mathsf{Cat}/X$$
$$x \mapsto x\backslash X,$$

where $x\backslash X$ is equipped with the canonical projection $x\backslash X \to X$.

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- Observation 2: the previous colimit is a *homotopy colimit* with respect to Thomason weak equivalences (follows from Thomason's theorem on homotopy colimits).

# Homology of the nerve as homotopy colimit

**Sketch of proof (continued):**

– Observation 3: the "functor" homology with local coefficient

$$H_*(N(-), M|_-): \mathsf{Cat}/X \to \mathrm{Der}(\mathsf{Ab}),$$

preserves homotopy colimits with respect to Thomason equivalences.

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Hence, the formula

$$H_*(N(X), M) \simeq \operatorname*{hocolim}_{x \in X^{\mathrm{op}}} H_*(N(x \backslash X), M|_{x \backslash X}).$$

# Homology of the nerve as homotopy colimit

**Sketch of proof (continued):**

– Final observation: The (nerve of the) category $x\backslash X$ is contractible, hence its homology is concentrated in degree 0 as

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– Final observation: The (nerve of the) category $x\backslash X$ is contractible, hence its homology is concentrated in degree 0 as

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and we conclude

$$H_*(N(X), M) \simeq \operatorname*{hocolim}_{x \in X^{\mathrm{op}}} H_*(N(x\backslash X), M|_{x\backslash X}) \simeq \operatorname*{hocolim}_{x \in X^{\mathrm{op}}} M_x.$$

□

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# Polygraphic homology as homotopy colimit

The goal now is to prove the following result.

**Proposition**

Let $X$ be a 1-category and $M\colon X^{\mathrm{op}} \to \mathrm{Ab}$ a local coefficient system on $X$. We have

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From that and what seen previously, we will deduce the desired result:

$$H_*^{\mathrm{pol}}(X, M) \simeq H_*(N(X), M).$$

# First step of the proof: the "unfolding" construction

Let $X$ be a 1-category and $p\colon P \to X$ a *polygraphic resolution*. By definition, we have

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We define a functor

$$\begin{aligned} X^{\mathrm{op}} &\to \omega\mathsf{Cat}/X \\ x &\mapsto x\backslash P, \end{aligned} \tag{2}$$

where $x\backslash P$ is defined as the pullback

$$\begin{array}{ccc} x\backslash P & \longrightarrow & P \\ {\scriptstyle x\backslash p}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ x\backslash X & \longrightarrow & X. \end{array}$$

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Observation: the colimit of (2) is $(P, p\colon P \to X)$.

# The key result

**Proposition (G., 2019)**

Let $X$ be a 1-category and $p \colon P \to X$ a polygraphic resolution. Then, the diagram

$$X^{\mathrm{op}} \to \omega\mathsf{Cat}$$
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is cofibrant for the projective folk model structure on $\mathrm{Hom}(X^{\mathrm{op}}, \omega\mathsf{Cat})$. In particular, every $x \backslash P$ is a polygraph.

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**Proof:** Not obvious.
Uses crucially the theory of discrete Conduché $\omega$-functors and the fact that

$$x\backslash P \to P$$

is such an $\omega$-functor for every $x$ in $X$.

# The key result

**Corollary**

The colimit of

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*homotopy* colimit (in the sense of the folk model structure).

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**Corollary**

The colimit of

$$X^{\mathrm{op}} \to \omega\mathsf{Cat}/X$$
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*homotopy* colimit (in the sense of the folk model structure).
Hence, we have

$$\operatorname*{hocolim}_{x \in X^{\mathrm{op}}} x\backslash P \simeq \operatorname*{colim}_{x \in X^{\mathrm{op}}} x\backslash P \simeq (P, p\colon P \to X)$$

in $\mathcal{H}\mathrm{o}(\omega\mathsf{Cat}^{\mathrm{folk}}/X)$

# A formula

Let $X$ be a 1-category, $M\colon X^{\mathrm{op}} \to \mathsf{Ab}$ a local coefficient system on $X$ and $p\colon P \to X$ a polygraphic resolution of $X$.

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Hence, we have the formula

$$H_*^{\mathrm{pol}}(X, M) \simeq H_*^{\mathrm{pol}}(P, M|_P) \simeq \operatorname*{hocolim}_{x \in X^{\mathrm{op}}} H_*^{\mathrm{pol}}(x\backslash P, M|_{x\backslash P}).$$

# The last piece of the puzzle

**Lemma**

For every object $x$ of $X$, the polygraphic homology of $x\backslash P$ with coefficient in $M|_{x\backslash P}$ is concentrated in degree 0 as:

$$H_*^{\mathrm{pol}}(x\backslash P, M|_{x\backslash P}) \simeq M_x.$$

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**Lemma**

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textbfIdea of proof:

- The 1-category $x\backslash X$ can be contracted to its initial object, meaning that

$$x\backslash X \underset{\mathrm{id}}{\overset{\mathrm{cst}}{\Downarrow}} x\backslash X.$$

# The last piece of the puzzle

**Idea of proof(continued)**:

- Because the canonical morphism $x\backslash P \to x\backslash X$ is a folk trivial fibration and $x\backslash P$ is cofibrant, we can lift the previous contraction to $x\backslash P$ :

$$x\backslash P \;\underset{\mathrm{id}}{\overset{\mathrm{cst}}{\Downarrow}}\; x\backslash P.$$

  where the 2-arrow is an *oplax* transformation.

- The polygraphic chain complex functor sends oplax transformation to chain complexes homotopy. Hence, the polygraphic homology of $x\backslash P$ is concentrated of degree 0 and its value is $M_x$. $\square$

# End of the proof of the main result

We can now finally conclude:

$$\begin{aligned} H_*^{\mathrm{pol}}(X, M) &\simeq \operatorname*{hocolim}_{x \in X^{\mathrm{op}}} H_*^{\mathrm{pol}}(x \backslash P, M|_{x \backslash P}) \\ &\simeq \operatorname*{hocolim}_{x \in X^{\mathrm{op}}} M_x \\ &\simeq H_*(N(X), M). \end{aligned}$$

Phew !

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- If we take *weak* polygraphic resolution, then would we have

$$H_*^{\mathrm{pol}}(X, M) \simeq H_*(N(X), M)\,?$$

Merci François !