Polygraphic homology of categories with local coefficients

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In the previous episode

Let $X$ be an $\omega$-category and $M: X^{\text{op}} \to \text{Ab}$ a local coefficient system on $X$. We have a canonical morphism

\[ H_*(N(X), M) \quad \longrightarrow \quad H^\text{pol}_*(X, M) \]

Intuition:

$\omega$-categories $\leftrightarrow$ spaces

Polygraphs $\simeq$ CW-complexes

Polygraphic homology $\simeq$ Cellular homology

Homology of the nerve $\simeq$ Singular homology

By analogy, (1) should be an isomorphism. But it is not in general.
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Homology of the nerve \quad Polygraphic homology

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Let $X$ be an $\omega$-category and $M : X^{\text{op}} \to \text{Ab}$ a local coefficient system on $X$. We have a canonical morphism

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\begin{align*}
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\end{align*}
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Let $X$ be an $\omega$-category and $M : X^{\text{op}} \to \text{Ab}$ a local coefficient system on $X$. We have a canonical morphism

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By analogy, (1) should be an isomorphism. But it is not in general.
The bubble

Counter-example:

\[ B := \alpha \]

We have \[ \text{Hol}(B, Z) = \begin{cases} Z & \text{if } k = 0, \\ 20 & \text{otherwise} \end{cases} \]

But \[ \text{N}(B) \] has the homotopy type of a \[ K(Z, 2) \], so \[ \text{Hol}(\text{N}(B), Z) = \begin{cases} Z & \text{if } k \text{ is even}, \\ 0 & \text{otherwise}. \end{cases} \]
Counter-example: Consider the “bubble”

\[ B := \bullet \]

\[ \alpha \]

\[ \overset{\text{H}_{\text{pol}}(B, Z)}{=} \begin{cases} Z & \text{if } k = 0, \\ 2Z & \text{otherwise} \end{cases} \]

\[ \overset{\text{H}_{\text{k}}(N(B), Z)}{=} \begin{cases} Z & \text{if } k \text{ is even}, \\ 0 & \text{otherwise} \end{cases} \]
Counter-example: Consider the “bubble”

\[ B := \begin{array}{c}
\alpha \\
\bullet
\end{array} \]

We have

\[ H_k^{pol}(B, \mathbb{Z}) = \begin{cases} 
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**Counter-example**: Consider the “bubble”\[B := \bullet\] We have\[H^\text{pol}_k(B, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0, 2 \\ 0 & \text{otherwise} \end{cases}\] But \(N(B)\) has the homotopy type of a \(K(\mathbb{Z}, 2)\), so\[H_k(N(B), \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k \text{ is even} \\ 0 & \text{otherwise} \end{cases}\]
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But the same equality also holds because of the Eckmann-Hilton argument

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Even though the bubble is a polygraph, in a “resolution” this last equality should only be an isomorphism. Then, in the “resolution”, there should be a 4-dimensional cell

\[ \text{id}_{\alpha \ast \alpha} \Rightarrow_4 \text{(EH)}. \]
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Then, in the “resolution”, there should be a 4-dimensional cell

\[ \text{id}_{\alpha \ast \alpha} \Rightarrow_4 (EH). \]

This cell generates non-trivial homology in dimension 4.
Goal of this talk

**Theorem (Maltsiniotis & G., 2023)**

Let $X$ be a 1-category and $M : X^{\text{op}} \to \text{Ab}$ a local coefficient system on $X$, then

$$H_* (N(X), M) \to H_*^{\text{pol}} (X, M)$$

is an isomorphism.

**Remark:** The (polygraphic) homology of a 1-category need not be trivial above dimension 1. This is because we have to take a polygraphic resolution $P \to X$, but $P$ is generally not a 1-category.
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For constant coefficient and $X$ monoid, this was proven by Lafont and Métayer (2009).
Theorem (Maltsiniotis & G., 2023)

Let $X$ be a 1-category and $M : X^{op} \to \text{Ab}$ a local coefficient system on $X$, then

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Yet another definition of homology

Recall that a local coefficient system on a 1-category $X$ is a functor $M : X^{\text{op}} \to \text{Ab}$ that sends all morphisms of $X$ to isomorphisms of $\text{Ab}$. 

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$$H_*^\text{loc}(X, M) := \text{ho colim}_{x \in X^{\text{op}}} M(x),$$

where $M$ is seen as a functor $X^{\text{op}} \to \text{Comp}(\text{Ab})$ concentrated in degree 0 and $\text{ho colim}$ is the left derived functor of the colimit.

Slogan: The homology is the derived colimit.
Yet another definition of homology

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**Definition**

Let $X$ be a 1-category and $M : X^{\text{op}} \to \text{Ab}$ a local coefficient system on $X$. We define the (total) homology of $X$ with coefficient in $M$ as the following object of $\text{Der}(\text{Ab})$

$$H_*(X, M) := \text{hocolim}_{x \in X^{\text{op}}} M_x,$$

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Slogan: The homology is the derived colimit.
Example:

Let $G$ be a group and $M : G^{\text{op}} \rightarrow \text{Ab}$ a right $G$-module. By definition

$$H_n(G, M) = \text{Tor}_n^{\mathbb{Z}[G]}(M, \mathbb{Z}),$$

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Hence, we recover

$$H_*(G, M) = \mathbb{L} \colim_{G^{\text{op}}} M =: \text{hocolim}_{G^{\text{op}}} M$$
The strategy is to use third definition of homology to connect the homology of the nerve with the polygraphic homology

\[ H_\ast(N(X), M) \simeq \text{hocolim} \ M_x \simeq H_\ast^{\text{pol}}(X, M). \]
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Homology of the nerve as homotopy colimit

**Proposition**

Let $X$ be a 1-category and $M : X^{\text{op}} \to \text{Ab}$ a local coefficient system on $X$. We have

$$H_*(N(X), M) \cong \text{hocolim}_{x \in X} M_x = H_*(X, M).$$
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**Sketch of proof:**

- Observation 1: $(X, 1_X)$ is the colimit of

$$X^{\text{op}} \to \text{Cat}/X$$

$$x \mapsto x \backslash X,$$

where $x \backslash X$ is equipped with the canonical projection $x \backslash X \to X$.

- Observation 2: the previous colimit is a homotopy colimit with respect to Thomason weak equivalences (follows from Thomason's theorem on homotopy colimits).
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Sketch of proof (continued):

- Observation 3: the “functor” homology with local coefficient

\[ H_\ast(N(\cdot), M|_{\cdot}) : \text{Cat}/X \to \text{Der}(\text{Ab}), \]

preserves homotopy colimits with respect to Thomason equivalences.
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**Sketch of proof (continued):**

- **Observation 3:** the "functor" homology with local coefficient

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Hence, the formula

\[ H_\ast(N(X), M) \simeq \text{hocolim}_{x \in X^{op}} H_\ast(N(x\setminus X), M\mid_{x\setminus X}). \]
Sketch of proof (continued):

- Final observation: The (nerve of the) category $x \setminus X$ is contractible, hence its homology is concentrated in degree 0 as

$$H_\ast(N(x \setminus X), M_{|x \setminus X}) \simeq M_x,$$
Sketch of proof (continued):

- Final observation: The (nerve of the) category $x \backslash X$ is contractible, hence its homology is concentrated in degree 0 as

$$H_*(N(x \backslash X), M|_{x \backslash X}) \simeq M_x,$$

and we conclude

$$H_*(N(X), M) \simeq \text{hocolim}_{x \in X^{op}} H_*(N(x \backslash X), M|_{x \backslash X}) \simeq \text{hocolim}_{x \in X^{op}} M_x.$$
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The goal now is to prove the following result.

**Proposition**

Let $X$ be a 1-category and $M : X^{\text{op}} \to \text{Ab}$ a local coefficient system on $X$. We have

$$H^\text{pol}_*(X, M) \simeq \text{hocolim}_{x \in X^{\text{op}}} M_x = H_*(X, M).$$
The goal now is to prove the following result.

**Proposition**

Let $X$ be a 1-category and $M : X^{\text{op}} \to \text{Ab}$ a local coefficient system on $X$. We have

$$H^\text{pol}_*(X, M) \simeq \text{hocolim}_{x \in X^{\text{op}}} M_x = H_*(X, M).$$

From that and what seen previously, we will deduce the desired result:

$$H^\text{pol}_*(X, M) \simeq H_*(N(X), M).$$
First step of the proof: the “unfolding” construction

Let $X$ be a 1-category and $p: P \to X$ a polygraphic resolution. By definition, we have

$$H_{\ast}^{\text{pol}}(X, M) \simeq H_{\ast}^{\text{pol}}(P, M)$$
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Let $X$ be a 1-category and $p: P \to X$ a polygraphic resolution. By definition, we have

$$H_{\ast}^{\text{pol}}(X, M) \simeq H_{\ast}^{\text{pol}}(P, M)$$

We define a functor

$$X^{\text{op}} \to \omega\text{Cat}/X$$

$$x \mapsto x \setminus P,$$

where $x \setminus P$ is defined as the pullback

$$\begin{array}{ccc}
  x \setminus P & \rightarrow & P \\
  x \setminus P & \downarrow & \downarrow p \\
  x \setminus X & \rightarrow & X.
\end{array}$$
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\[ \begin{array}{ccc}
   x \backslash P & \rightarrow & P \\
   \downarrow & & \downarrow p \\
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\end{array} \]

Observation: the colimit of (2) is $(P, p: P \rightarrow X)$. 
Proposition (G., 2019)

Let $X$ be a 1-category and $p: P \to X$ a polygraphic resolution. Then, the diagram

$$
\begin{align*}
X^{\text{op}} & \to \omega \text{Cat} \\
\downarrow & \\
\omega \text{Cat} & \to \\
\end{align*}
$$

$x \mapsto x|P$

is cofibrant for the projective folk model structure on $\text{Hom}(X^{\text{op}}, \omega \text{Cat})$. In particular, every $x|P$ is a polygraph.
The key result

Proposition (G., 2019)

Let $X$ be a 1-category and $p: P \rightarrow X$ a polygraphic resolution. Then, the diagram

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**Proof:** Not obvious.
The key result

**Proposition (G., 2019)**

Let $X$ be a 1-category and $p: P \rightarrow X$ a polygraphic resolution. Then, the diagram

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$$x \mapsto x\backslash P$$

is cofibrant for the projective folk model structure on $\text{Hom}(X^{\text{op}}, \omega\text{Cat})$. In particular, every $x\backslash P$ is a polygraph.

**Proof:** Not obvious. Uses crucially the theory of discrete Conduché $\omega$-functors and the fact that

$$x\backslash P \rightarrow P$$

is such an $\omega$-functor for every $x$ in $X$. 
The key result

Corollary

The colimit of

\[ X^{\text{op}} \to \omega \text{Cat}/X \]
\[ x \mapsto x \setminus P, \]

_homotopy_ colimit (in the sense of the folk model structure).
The key result

**Corollary**

The colimit of

\[ X^{\text{op}} \to \omega\text{Cat}/X \]

\[ x \mapsto x\backslash P, \]

*homotopy* colimit (in the sense of the folk model structure).

Hence, we have

\[ \text{hocolim}_{x \in X^{\text{op}}} x\backslash P \simeq \text{colim}_{x \in X^{\text{op}}} x\backslash P \simeq (P, p: P \to X) \]

in \( \mathcal{H}_0(\omega\text{Cat}^{\text{folk}}/X) \)
Let $X$ be a 1-category, $M: X^{\text{op}} \rightarrow \text{Ab}$ a local coefficient system on $X$ and $p: P \rightarrow X$ a polygraphic resolution of $X$. 

Observation: The polygraphic homology functor $H_{\text{pol}}^\ast(\cdot, M|\cdot): \omega\text{Cat}/X \rightarrow \text{Der}(\text{Ab})$ preserves homotopy colimits with respect to the folk model structure. Hence, we have the formula:

$$H_{\text{pol}}^\ast(X, M) \cong H_{\text{pol}}^\ast(P, M|P) \cong \text{hocolim}_{x \in X^{\text{op}}} H_{\text{pol}}^\ast(x|P, M|x|P).$$
Let $X$ be a 1-category, $M : X^{\text{op}} \to \text{Ab}$ a local coefficient system on $X$ and $p : P \to X$ a polygraphic resolution of $X$.

Observation: The polygraphic homology “functor”

$$H^\text{pol}_* (\_ , M|_{\_} ) : \omega \text{Cat}/X \to \text{Der}(\text{Ab})$$

preserves homotopy colimits with respect to the folk model structure.
Let $X$ be a 1-category, $M : X^{\text{op}} \to \text{Ab}$ a local coefficient system on $X$ and $p : P \to X$ a polygraphic resolution of $X$.

**Observation:** The polygraphic homology “functor”

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H^\text{pol}_*(-, M|_-) : \omega\text{Cat}/X \to \text{Der}((\text{Ab})
\]

preserves homotopy colimits with respect to the folk model structure. Hence, we have the formula

\[
H^\text{pol}_*(X, M) \simeq H^\text{pol}_*(P, M|_P) \simeq \underset{x \in X^{\text{op}}}{\text{hocolim}} H^\text{pol}_*(x \setminus P, M|_{x \setminus P}).
\]
Lemma

For every object $x$ of $X$, the polygraphic homology of $x \setminus P$ with coefficient in $M|_{x \setminus P}$ is concentrated in degree 0 as:

$$H^\text{pol}_\ast(x \setminus P, M|_{x \setminus P}) \simeq M_x.$$
The last piece of the puzzle

Lemma

For every object $x$ of $X$, the polygraphic homology of $x \setminus P$ with coefficient in $M|_{x \setminus P}$ is concentrated in degree 0 as:

$$H_*^{\text{pol}}(x \setminus P, M|_{x \setminus P}) \simeq M_x.$$ 

Idea of proof:

- The 1-category $x \setminus X$ can be contracted to its initial object, meaning that

$$x \setminus X \xrightarrow{\text{id}} x \setminus X \xrightarrow{\text{cst}} x \setminus X.$$
The last piece of the puzzle

Idea of proof (continued):

− Because the canonical morphism $x\backslash P \to x\backslash X$ is a folk trivial fibration and $x\backslash P$ is cofibrant, we can lift the previous contraction to $x\backslash P$:

$$
\begin{tikzcd}
x\backslash P & x\backslash P \\
\downarrow & \downarrow \\
x\backslash P & x\backslash P.
\end{tikzcd}
$$

where the 2-arrow is an oplax transformation.

− The polygraphic chain complex functor sends oplax transformation to chain complexes homotopy. Hence, the polygraphic homology of $x\backslash P$ is concentrated of degree 0 and its value is $M_x$. □
We can now finally conclude:

\[ H^\text{pol}_*(X, M) \cong \text{hocolim}_{x \in X^\text{op}} H^\text{pol}_*(x \setminus P, M|_{x \setminus P}) \]

\[ \cong \text{hocolim}_{x \in X^\text{op}} M_x \]

\[ \cong H_*(N(X), M). \]

Phew!
Perspective

− Could we make sense of the formula $H^\ast(X, M) = \operatorname{holim}_{x \in X} M_x$ when $X$ is an $\omega$-category?

− Would we still have $H^\ast(N(X), M) = \operatorname{holim}_{x \in X} M_x$?

− If we take weak polygraphic resolution, then would we have $H_{\text{pol}}^\ast(X, M) \simeq H^\ast(N(X), M)$?
Could we make sense of the formula

$$H_\ast(X, M) = \text{hocolim}_{x \in X^{\text{op}}} M_x$$

when $X$ is an $\omega$-category?
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\[ H_*(X, M) = \underset{x \in X^{op}}{\text{hocolim}} M_x \]

when \( X \) is an \( \omega \)-category?

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Could we make sense of the formula
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\[ H_*(N(X), M) = \operatorname{hocolim}_{x \in X^{\text{op}}} M_x \]?

If we take weak polygraphic resolution, then would we have
\[ H^\text{pol}_*(X, M) \simeq H_*(N(X), M) \]?
Merci François !