The Cyclic Category Revisited

André Joyal and Amit Sharma

Colloque en l'honneur de François Metayer (Paris, 8/06/2023)

Joyeux anniversaire François!

Je voudrais te remercier pour tes belles contributions aux catégories supérieures et à l'informatique théorique !

Je souligne particulièrement ta contribution à l'incroyable structure folk sur ω-Cat !

Le séminaire Métayer joue un rôle important dans la diffusion et la communication des travaux de toute une communauté de chercheurs.

Merci encore!

The cyclic category Λ has 40 years!

- A. Connes: *Cohomology cyclique et foncteurs* Ext^n. (1983).
- W.G. Dwyer, M. J. Hopkins and D.M. Kan: *The homotopy theory of cyclic sets*. (1985).
- A. Connes and C. Consani: *Cyclic structures on the topos of cyclic sets*. (2013).
- O. Caramello and N. Wentzlaff: *Cyclic theories*. (2014).
- T. Nikolaus and P. Scholze: *On topological Cyclic Homology*. (2018).
- D. Kaledin: *Trace theories, Böksted periodicity and Bott periodicity*. (2021).
- K. Hess and N. Rasekh: *Shadows and bicategorical traces*. (2023).

Eckmann-Hilton

Theorem

(Eckmann-Hilton) *Let* $\mathcal{M} = (\mathcal{M}, \otimes, I)$ *be a monoidal category. The monoid of endomorphisms of the unit object* $I \in \mathcal{M}$ *is commutative and we have* $f \circ g = f \otimes g$ *for every* $f, g \in End(I)$*.*

For example, the category $[\mathbb{C}, \mathbb{C}]$ of endo-functors of a category $\mathbb{C}$ is monoidal if we put $F \otimes G = F \circ G$ for $F, G : \mathbb{C} \to \mathbb{C}$. The unit object $I \in [\mathbb{C}, \mathbb{C}]$ is the identity functor $1_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}$.

An element $\alpha \in End(1_{\mathbb{C}})$ is a natural transformation $\alpha : 1_{\mathbb{C}} \to 1_{\mathbb{C}}$. By naturality, the following square commutes for every map $f : A \to B$ in $\mathbb{C}$.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \alpha_A}\downarrow & & \downarrow{\scriptstyle \alpha_B} \\ A & \xrightarrow{f} & B \end{array}$$

Hence we have $f\alpha_A = \alpha_B f$ for every map $f : A \to B$.

BM-categories

If M is a monoid, we shall denote by BM the category with a single object $\star$ and with $Hom(\star, \star) = M$.

The category BM is symmetric monoidal when M is commutative. We have $f \otimes g = f \circ g$ for every $f, g \in M$.

Definition

A *BM-category* is a category $\mathbb{C}$ equipped with an action

$$\alpha : BM \times \mathbb{C} \to \mathbb{C}$$

of the monoidal category BM.

An action $\alpha : BM \times \mathbb{C} \to \mathbb{C}$ is equivalent to a morphism of monoids

$$\alpha : M \to End(1_{\mathbb{C}})$$

BM-categories

Remarks:

- the category of M-sets $Set^M = [BM, Set]$ is symmetric monoidal closed, with the tensor product given by the Day convolution product;
- A BM-category is the same thing as a category $\mathbb{C}$ enriched over the symmetric monoidal category Set^M.

The tensor product $X \otimes_M Y$ of two M-sets X and Y is the coequaliser of the maps

$$d_0, d_1 : X \times M \times Y \to X \times Y$$

defined by $d_0(x, m, y) = (m \cdot x, y)$ and $d_1(x, m, y) = (x, m \cdot y)$.

The action of M on $X \otimes_M Y$ is defined by putting $m \cdot (x \otimes y) = (m \cdot x) \otimes y = x \otimes (m \cdot y)$.

The category $B\mathbb{N}$

The additive monoid of natural numbers $\mathbb{N} = (\mathbb{N}, +, 0)$ is freely generated by $1 \in \mathbb{N}$.

A $B\mathbb{N}$-category is a category $\mathbb{C}$ equipped with a homomorphism

$$\alpha : \mathbb{N} \to End(1_{\mathbb{C}})$$

Equivalently, $B\mathbb{N}$-category is a category $\mathbb{C}$ equipped with a natural transformation $\rho = \alpha(1) : 1_{\mathbb{C}} \to 1_{\mathbb{C}}$.

We shall say that the endomorphism $\rho_A : A \to A$ is the *structural loop* of the object $A \in \mathbb{C}$.

The naturality of ρ means that we have $f\rho_A = \rho_B f$ for every map $f : A \to B$ in $\mathbb{C}$.

A functor $F : \mathbb{A} \to \mathbb{B}$ between $B\mathbb{N}$-categories is *equivariant* if $F(\rho_A) = \rho_{FA}$ for every object $A \in \mathcal{A}$.

The category Δ

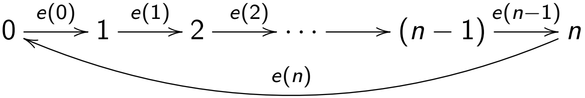
Recall that the *simplicial category* Δ has for objects the non-empty finite ordered sets $[n] = \{0, 1, \ldots, n\}$ for $n \geq 0$ and for morphisms the order preserving maps.

The category of posets is a full subcategory of the category of (small) categories and we have $\Delta \subset Poset \subset Cat$.

The category $[n]$ is freely generated by a chain of n-arrows

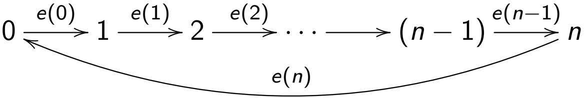
$$0 \to 1 \to 2 \to \cdots \to n-1 \to n$$

We shall denote by $[n]^+$ the category freely generated by a circuit of $n+1$-arrows



We have $e(i) : i \to (i+1)$ for every $i \in \mathbb{Z}/(n+1)$.

The category $[n]^+$



The monoid of endomorphisms $End(i) = Hom(i, i)$ of the object $i \in [n]^+$ is freely generated by the loop

$$\rho(i) := e(i+n) \cdots e(i+1)e(i)$$

We have $f\rho(i) = \rho(j)f$ for every morphism $f : i \to j$ in $[n]^+$.

Hence the category $[n]^+$ has a $B\mathbb{N}$-structure defined by the loops $\rho(i) : i \to i$ for every $i \in [n]^+$.

Notice that $[0]^+ = B\mathbb{N}$.

The category Λ

Definition

We shall say that a functor $f : [m]^+ \to [n]^+$ has *degree* $q \geq 0$ if $f(\rho_i) = (\rho_{f(i)})^q$ for every $i \in [m]$.

Remarks

- If $f(\rho_i) = (\rho_{f(i)})^q$ for some $i \in [m]$, then f has degree q.
- A functor $f : [m]^+ \to [n]^+$ is a $B\mathbb{N}$-functor if and only it is of degree 1.

Definition

The *epicyclic category* $\tilde{\Lambda}$ is the subcategory of *Cat* spanned by the categories $[n]^+$ and the functors of degree > 0.

Definition

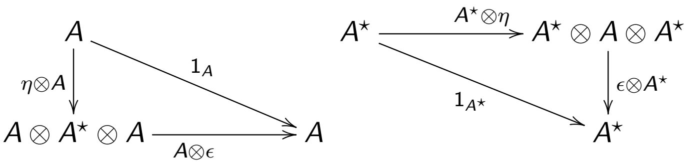
The *cyclic category* Λ is the subcategory of *Cat* spanned by the categories $[n]^+$ and the functor of degree 1.

The cyclic category Λ is a full subcategory of the category of $B\mathbb{N}$-categories.

Dual objects

Let $\mathcal{M} = (\mathcal{M}, \otimes, I, c)$ be a symmetric monoidal category.

A duality $(\epsilon, \eta) : A^\star \dashv A$ between two objects A and $A^\star$ in $\mathcal{M}$ is a pair of maps $\epsilon : A^\star \otimes A \to I$ and $\eta : I \to A \otimes A^\star$ such that the following diagrams commute:



In which case the functor $A^\star \otimes - : \mathcal{M} \to \mathcal{M}$ is left adjoint to the functor $A \otimes - : \mathcal{M} \to \mathcal{M}$. Moreover,

$$(\epsilon, \eta) : A^\star \dashv A \quad \Leftrightarrow \quad (\epsilon\sigma, \sigma\eta) : A \dashv A^\star$$

where $\sigma : A \otimes A^\star \to A^\star \otimes A$ is the symmetry.

Compact closed categories and traces

A symmetric monoidal category $\mathcal{M} = (\mathcal{M}, \otimes, I, c)$ is said to be *compact closed* if every object $A \in \mathcal{M}$ has a dual $A^\star := (A^\star, \epsilon, \eta)$.

A compact closed category $\mathcal{M}$ has an internal hom $[A, B] := A^\star \otimes B$. Every map $f : A \to B$ is represented by an element $\ulcorner f \urcorner : I \to [A, B]$.

The *trace map* $tr_A : [A, A] \to I$ is the co-unit $\epsilon : A^\star \otimes A \to I$.

The *trace* of an endomorphism $f : A \to A$ is defined to be the composite

$$Tr_A(f) = tr \circ \ulcorner f \urcorner : I \to I$$

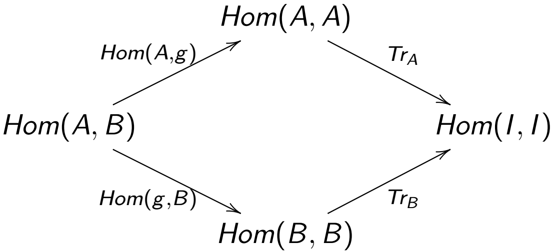
This defines a map

$$Tr_A : Hom(A, A) \to Hom(I, I)$$

Trace and co-end

If $f : A \to B$ and $g : B \to A$, then $Tr_A(gf) = Tr_B(fg)$.

Hence the following square commutes for every map $g : B \to A$,



The induced map

$$\int^{A \in \mathcal{M}} Hom(A, A) \longrightarrow Hom(I, I)$$

is bijective.

Free compact closed categories I

We shall denote by $[\mathcal{E}, \mathcal{F}]$ the category of functors between two categories, and denote its sub-groupoid isomorphisms by $[\mathcal{E}, \mathcal{F}]^{\simeq}$

We shall denote by $SMon[\mathcal{M}, \mathcal{N}]$ the category of symmetric monoidal functors (and symmetric monoidal transformations) between two symmetric monoidal categories, and denote its sub-groupoid of isomorphisms by $SMon[\mathcal{M}, \mathcal{N}]^{\simeq}$.

Definition

If $\mathbb{A}$ is a category we shall say that a functor $u : \mathbb{A} \to \mathcal{C}(\mathbb{A})$ *exhibits the compact closed category freely generated by* $\mathbb{A}$ if the category $\mathcal{C}(\mathbb{A})$ is a compact closed and the restriction functor

$$u^{\star} : SMon[C(\mathbb{A}), \mathcal{M}]^{\simeq} \to [\mathbb{A}, \mathcal{M}]^{\simeq}$$

is an equivalence of groupoids for every compact closed category $\mathcal{M}$.

Free compact closed categories II

A variant of the following theorem was proved by Kelly-Laplaza (1980):

Theorem

[Kelly-Laplaza, Sharma] *Every small category* $\mathbb{A}$ *generates freely a compact closed category* $u : \mathbb{A} \to \mathcal{C}(\mathbb{A})$.

The *Kelly-Lapaza trace* of a category $\mathbb{A}$ is defined by putting

$$KLT(\mathbb{A}) := \int^{A \in \mathbb{A}} \mathbb{A}(A, A).$$

Theorem

[Kelly-Laplaza] *If* $\mathbb{A}$ *is a category, then the commutative monoid* $\mathcal{C}(\mathbb{A})(I, I)$ *is freely generated by* $KLT(\mathbb{A})$.

The 2-category of distributors

Recall that a *distributor* $M : \mathbb{A} \Rightarrow \mathbb{B}$ between two small categories $\mathbb{A}$ and $\mathbb{B}$ is defined to be a functor $M : \mathbb{A}^{op} \times \mathbb{B} \to Set$.

Remark: A distributor $M : \mathbb{A} \Rightarrow \mathbb{B}$ is equivalent to a cocontinuous functor

$$[M] : [\mathbb{A}, Set] \to [\mathbb{B}, Set]$$

The composite $N \circ M$ of two distributors $M : \mathbb{A} \Rightarrow \mathbb{B}$ and $N : \mathbb{B} \Rightarrow \mathbb{C}$ is defined by putting

$$(N \circ M)(A, C) = \int^{B \in \mathbb{B}} M(A, B) \times N(B, C)$$

The identity distributor $1_{\mathbb{A}} : \mathbb{A} \Rightarrow \mathbb{A}$ is the hom functor

$$\mathbb{A}(-, -) : \mathbb{A}^{op} \times \mathbb{A} \to Set$$

The 2-category of distributors is symmetric monoidal closed

The tensor product of two distributors $M : \mathbb{A} \Rightarrow \mathbb{B}$ and $N : \mathbb{C} \Rightarrow \mathbb{D}$ is the distributor $M \boxtimes N : \mathbb{A} \times \mathbb{C} \Rightarrow \mathbb{B} \times \mathbb{D}$ defined by letting

$$(M \boxtimes N)(A \boxtimes C, B \boxtimes D) := M(A, B) \times N(C, D)$$

for every $A \boxtimes C := (A, C) \in \mathbb{A} \times \mathbb{C}$ and $B \boxtimes D := (B, D) \in \mathbb{B} \times \mathbb{D}$.

The *dual* of a small category $\mathbb{A}$ is its opposite $\mathbb{A}^{op}$.

The unit $\eta : 1 \Rightarrow \mathbb{A} \times \mathbb{A}^{op}$ is the functor $\eta : \mathbb{A} \times \mathbb{A}^{op} \to Set$ defined by letting $\eta(A \boxtimes B^{o}) := \mathbb{A}(B, A)$.

The counit $\epsilon : \mathbb{A}^{op} \times \mathbb{A} \Rightarrow 1$ is the functor $\epsilon : (\mathbb{A}^{op} \times \mathbb{A})^{op} \to Set$ defined by letting $\epsilon((A \boxtimes B^{o})^{o}) := \mathbb{A}(B, A)$

The trace of a distributor is a coend

The trace of a distributor $M : \mathbb{A} \Rightarrow \mathbb{A}$ is equal to its coend

$$Tr(M) = \int^{A \in \mathbb{A}} M(A, A)$$

The KL-trace of $\mathbb{A}$ is the trace of the identity distributor $1_{\mathbb{A}} : \mathbb{A} \Rightarrow \mathbb{A}$

$$KLT(\mathbb{A}) := \int^{A \in \mathbb{A}} \mathbb{A}(A, A) =: Tr(1_{\mathbb{A}})$$

Compact closed ∞-categories

Here, ∞-categories $=$ quasi-categories

The following notions were extended from categories to quasicategories by Lurie [HA].

- Symmetric monoidal categories $\to$ symmetric monoidal quasicategories
- Dual object in a category $\to$ dual object in a quasicategory
- Compact closed categories $\to$ compact closed quasicategories
- Free compact closed categories $\to$ free compact closed quasicategories

The ∞-category Bord_1

An object of the ∞-category Bord_1 is a compact oriented 0-manifold (=finite signed set), and a morphisms $A \to B$ is a compact oriented 1-manifold with boundary M such that $\partial M = -A \sqcup B$.

We refer to [Lur] for a description of the higher cells of the quasi-category Bord_1.

The ∞-category Bord_1 is symmetric monoidal closed:

- $A \otimes B := A \sqcup B$
- $M \otimes N := M \sqcup N$

The unit object is the emptyset.

1-bordism

Composition of 1-bordisms

The space $\mathrm{Bord}_1(\emptyset, \emptyset)$

The space $\mathrm{Bord}_1(\emptyset, \emptyset)$ is the classifying space of compact oriented 1-manifolds without boundary.

Recall that every compact oriented 1-manifold without boundary is a finite disjoint union of oriented circles S^1.

The classifying space of compact *connected* oriented 1-manifolds without boundary is $BDiff^+(S^1)$, where $Diff^+(S^1)$ is the topological group of orientation preserving diffeomorphims of S^1.

The inclusion $SO(2) \subseteq Diff^+(S^1)$ is a homotopy equivalence and $BSO(2) = K(\mathbb{Z}, 2) = \mathbb{C}P^\infty$.

It follows that

$$\mathrm{Bord}_1(\emptyset, \emptyset) = \coprod_{n \geq 0} (\mathbb{C}P^\infty)^n /\!\!/ \Sigma_n$$