Resolutions and abstract abstract coherence

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- **Introductory part and motivations:**
  from syzygies to resolutions by polygraphs

- **Main part: Abstract abstract coherence**
  (joint work with Georg Struth, Cameron Calk and Eric Goubault)

Introductory part and motivations: from syzygies to resolutions by polygraphs

From syzygies to resolutions

- A **syzygy** is a relation between generators of a module.
  - ▷ From Latin **syzygia** and Greek $\sigma\upsilon\zeta\upsilon\gamma\iota\alpha$: union, conjunction, yoked together.

- Given a finitely generated module M on a commutative ring R and a set of generators:

$$Y = \{\mathbf{y}_1, \ldots, \mathbf{y}_k\}$$

▷ a **syzygy** of M is an element $(\lambda_1, ..., \lambda_k)$ in R^k for which

$$\lambda_1 \mathbf{y}_1 + \ldots + \lambda_k \mathbf{y}_k = 0$$

▷ The set of all syzygies wrt Y is a submodule of R^n called the **module of first syzygies**.

▷ The **second syzygy module** is the module of the relations between generators of the first syzygy module.

▷ In this way, for any $n \geqslant 2$, one defines, the **nth syzygy module**.

Theorem. (Hilbert's Syzygy Theorem, 1890)
If M is a finitely generated module over the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$, then the nth syzygy module of M is a always a free module.

- This implies that M has a finite free resolution of length at most n.

Resolutions and linear rewriting

- A good algorithmic way to calculate syzygies over a commutative ring are **Gröbner basis algorithms**.

- In commutative algebra, this approach was progressively formalized throughout the twentieth century.

  - L.E. Dickson 1913, N. Günther 1913, F. S. Macaulay 1916, M. Janet 1920, E. Noether 1921, G. Hermann 1926, W. Gröbner 1937, B. Buchberger 1965...

  - F.-O. Schreyer, 1980 : computation of syzygies by means of the **division algorithm**.
    - Buchberger's completion algorithm computes **Gröbner bases**.
    - The reduction to zero of a **S-polynomial** in a Gröbner basis gives a syzygy.

- Other approaches to the notion of Gröbner basis:
  - A. Shirshov, 1962: **Composition Lemma** for Lie algebras,
  - H. Hironaka, 1966: **Standard basis** for power series rings,
  - L. Bokut, 1976, G. Berman, 1978: **Composition Lemma** and **Diamond Lemma** for associative algebras

- **Resolutions for associative algebras** using Gröbner bases, D. J. Anick, 1986, D. J. Anick - E. L. Green, 1987.
- **Resolutions for monoids** using String rewriting, K. S. Brown, 1992, Y. Kobayashi, 1990, J.R.J. Groves, 1990.

Resolutions and linear rewriting

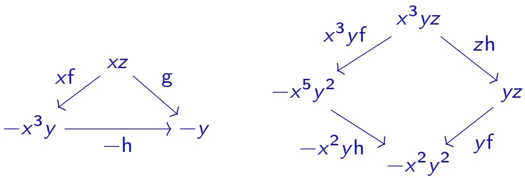
- $I = \langle f, g \rangle \subset \mathbb{K}[x, y, z]$, with

$$f = x^2y + z, \qquad g = xz + y.$$

- With respect the monomial order $\prec_{lex}$, with $x < y < z$ we have a Gröbner basis for I:

$$z \xrightarrow{\mathrm{f}} -x^2y, \qquad xz \xrightarrow{\mathrm{g}} -y, \qquad x^3y \xrightarrow{\mathrm{h}} y.$$

- Two critical branchings:



- Syzygies

$$\mathrm{g} = x\mathrm{f} - \mathrm{h} \qquad\qquad (z + x^2y)\mathrm{h} = (x^3y - y)\mathrm{f}$$

$$\boxed{(z + x^2y)\mathrm{g} + (-zx - y)\mathrm{f} \;= 0}$$

Syzygies by categories and Syzygies for categories

Journal of Pure and Applied Algebra 49 (1987) 201–217
North-Holland

WORD PROBLEMS AND A HOMOLOGICAL FINITENESS CONDITION FOR MONOIDS

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Introduction

Our purpose is to prove that a monoid which has a 'nice' solution to its word problem satisfies a certain homological finiteness condition. More precisely, we prove: if a monoid S has a finite terminating Church–Rosser presentation, then S is $(\mathrm{FP})_3$; this is Theorem 4.1 below. (See Section 2 for the definition of "terminating" and "Church–Rosser".) Examples of groups that are not $(\mathrm{FP})_3$ are known; see Section 4 for a brief description of several of these. For completeness, we provide an example of a monoid that is not $(\mathrm{FP})_3$. In each case, the monoid (or group) is finitely-presented and has a solvable word problem. These examples answer (in the negative) the following question of Jantzen [15]: does a finitely-presented monoid with a solvable word problem have a finite terminating Church–Rosser presentation?

The Church–Rosser property was discovered by Church and Rosser [9] during the course of research on the Λ-calculus. Properties of terminating relations were investigated by Newman [16]. For a systematic treatment of both topics together with further references, see [14]. Monoids with terminating Church–Rosser presentations have been studied by Nivat [17] and others. See [5] for a recent survey.

We conclude this introduction with a brief outline of what follows and some further discussion.

Section 1 contains basic results on Noetherian relations. In particular, we develop some tools for dealing with free abelian groups which have a basis ordered by a Noetherian relation.

Section 2 introduces terminating and Church–Rosser presentations. (Because of difficulties in verifying that the relation $\rightarrow$ defined in Section 2 is Noetherian, it is common to assume that the rewriting rules R are length-reducing: if $(r, s) \in R$, then $|r| > |s|$. We specifically do not make this assumption, so that our terminology differs, for example, from that of [5].) Variations of Theorem 2.1, which gives

Theoretical Computer Science 115 (1993) 43–62
Elsevier

Higher-dimensional word problems with applications to equational logic

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Abstract

Burroni, A., Higher-dimensional word problems with applications to equational logic, Theoretical Computer Science 115 (1993) 43–62.

In this paper we reduce equational logic to a two-dimensional word problem (Theorem 2.3) and introduce the concept of an n-dimensional word problem for all $n \in \mathbb{N}$, with an emphasis on geometrical meaning.

The word problem on a monoid admits two natural generalizations:

- The first one is the extension from monoids to categories. In this case, the words become "paths" in a graph, and the equality of paths is a problem of commutation of diagrams.
- The second one is the extension from monoids to universal algebras. In this case, the words become "terms", and the word problem becomes derivation in equational logic from given equations.

It is possible to unify these two generalizations?

In this paper, we answer as follows: the latter problem is nothing but a 2-dimensional word problem in a "2-monoid", which leads to the syntactical study of a 3-category. This crucial observation leads to the general problem for n-paths in an n-category, or even in an ∞-category. A lot of computations made by category theorists are 1-, 2- or 3-dimensional; in fact, n-dimensional computations take place in an $(n+1)$-category. Furthermore, beyond the unity thus given to various Thue problems, the link with combinatorial topology appears, rewriting systems being in this setting a refinement of homotopy theory.

Some ideas of this paper, which is an extended version of [4] have their origin in the dimensional analysis of formal languages [2] and the "elimination" of the universal property of cartesian product [3]. Theorem 2.2 was first communicated in March

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Syzygies by categories and Syzygies for categories

- Two seminal results on syzygies by and for categories.

- C.C. Squier, 1987.

Theorem 4.1. *If a monoid S has a finite terminating Church–Rosser presentation, then S is* $(\mathrm{FP})_3$.

  - Construction of (abelian) resolutions for monoids using string rewriting.

- A. Burroni, 1993.

Theorem 2.3. *For all finite* (Ω, E), *the 2-monoid* $T = \boldsymbol{T}(\Omega, E)$ *is finitely presented* (*i.e. it has a finite CW-presentation in the sense of Section* 1.2).

  - Equational presentations of Lawvere theories using 3-polygraphs.

Résolutions by polygraphs

Theory and Applications of Categories, Vol. 11, No. 7, 2003, pp. 148–184.

RESOLUTIONS BY POLYGRAPHS

FRANÇOIS MÉTAYER

ABSTRACT. A notion of resolution for higher-dimensional categories is defined, by using polygraphs, and basic invariance theorems are proved.

1. Introduction

Higher-dimensional categories naturally appear in the study of various rewriting systems. A very simple example is the presentation of $\mathbf{Z}/2\mathbf{Z}$ by a generator a and the relation $aa \to 1$. These data build a 2-category X :

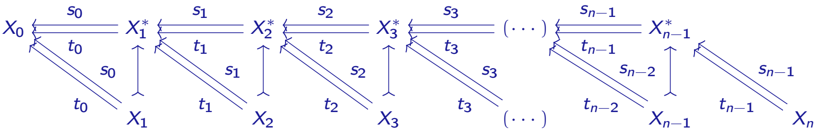
$$X_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} X_1 \underset{t_1}{\overset{s_1}{\leftleftarrows}} X_2$$

where $X_0 = \{\bullet\}$ has a unique 0-cell, $X_1 = \{a^n/n \geq 0\}$ and X_2 consists of 2-cells $a^n \to a^p$, corresponding to different ways of rewriting a^n to a^p by repetitions of $aa \to 1$, up to suitable identifications. 1-cells compose according to $a^n *_0 a^p = a^{n+p}$, and 2-cells compose vertically, as well as horizontally, as shown on Figure 1, whence the 2-categorical structure on X.

▶ An ***n*-polygraph** is a sequence

$$X = (X_0, X_1, \dots, X_n)$$

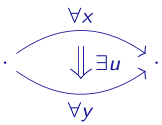
constructed by induction

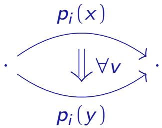


Résolutions by polygraphs

- An ω-functor $p : C \to D$ is an **acyclic fibration** if $p_0 : C_0 \to D_0$ is onto and p has the lifting property:

  for any i-cells $x || y$ in C and for any $v : p_i(x) \to p_i(y)$ in D, there is some $u : x \to y$ in C such that $p_{i+1}(u) = v$.





- A **polygraphic resolution** of an ω-category C is an acyclic fibration

$$p : X^* \to C$$

where X^* is a free ω-category on an ω-polygraph X.

Resolutions by polygraphs

Theorem. (Métayer 2003)

▷ Any ω-category C has a polygraphic resolution.

▷ Given two such polygraphic resolutions $p : X^* \to C$ and $q : Y^* \to C$, there is some ω-functor $F : X^* \to Y^*$ such that the following diagram commutes:

$$\begin{array}{ccc} X^* & \xrightarrow{F} & Y^* \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & C & \end{array}$$

▷ For any two such ω-functors $F, G : X^* \to Y^*$, we get a homotopy $\xi : F \to G$.

Consequences.

▶ Any two polygraphic resolutions of an ω-category C are homotopically equivalent.

▶ The **polygraphic homology** of an ω-category C is

$$\mathrm{H}_*^{\mathrm{pol}}(C) := \mathrm{H}_*(\mathbb{Z}X)$$

where $X^* \to C$ is a polygraphic resolution of C.

▶ (Lafont-Métayer, 2009) For a monoid M, $\mathrm{H}_*^{\mathrm{pol}}(M) \simeq \mathrm{H}_*(M, \mathbb{Z})$.

Problems on polygraphic resolutions

- Three lines of research (Métayer-Lafont, 2009)

Problem A.

A general finiteness conjecture (Lafont, 2007): is it true that a monoid M presented by a finite, terminating and confluent rewriting system has a polygraphic resolution

$$X^* \to M$$

where X_i is finite in each dimension?

Problem B.

How to define a notion of polygraphic resolution for other structures expressible by polygraphs (proofs systems, Petri nets, term algebras...)?

Problem C.

Are there any applications to the theory of directed homotopy?

Problems on polygraphic resolutions

Problem.

How can polygraphic resolutions be algebraically formulated with a view to formalization in proof assistants?

Issues.

- Algebraisation of the structure of polygraphs (higher dimensional rewriting system) and their properties:
  - Abstraction of diagrammatic reasoning: confluence, termination...
  - Homotopical properties: acyclicity, contracting homotopies, normalisation strategies...
- The algebraisation of the calculation of syzygies by rewriting.
  - Church-Rosser, Newman, and Squier machineries...
- The formalisation in proof assistants.
  - Isabelle...

Main part:
Abstract abstract coherence

This is a joint work with

Georg Struth, Cameron Calk and Eric Goubault

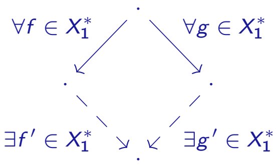
Plan

- Part I: Confluence proofs in modal Kleene algebras
- Part II: Abstract coherence by rewriting
- Part III: Coherent proofs in higher modal Kleene algebras
- Conclusion: Work in progress

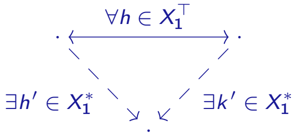
Part I:
Calculating confluence proofs
in modal Kleene algebras

Church-Rosser Theorem (diagrammatic formulation)

- An **abstract rewriting system** is a 1-polygraph (X_0, X_1)
  - It is **confluent** if



  - It has the **Church-Rosser property** if



Theorem. (Church-Rosser, 1936)
A 1-polygraph is confluent if and only if it is Church-Rosser.

Church-Rosser Theorem (relational formulation)

- An **abstract rewriting system** on a set X is a binary relation $\to$ on X
  - It is **confluent** if
    $$\leftarrow^* \cdot \to^* \subseteq \to^* \cdot \leftarrow^*$$
    - where $\to^*$ denotes the reflexive, transitive closure of $\to$
    - and $\leftarrow$ its converse, and $\cdot$ the relational composition.
  - It has the **Church-Rosser property** if
    $$(\to \cup \leftarrow)^* \subseteq \to^* \cdot \leftarrow^*$$

where $(\to \cup \leftarrow)^*$ is the reflexive, symmetric and transitive closure of $\to$.

Theorem. (Church-Rosser, 1936)

$$\left(\leftarrow^* \cdot \to^* \subseteq \to^* \cdot \leftarrow^* \right) \quad \Leftrightarrow \quad \left((\to \cup \leftarrow)^* \subseteq \to^* \cdot \leftarrow^* \right)$$

Church-Rosser Theorem (algebraic formulation)

- A **semiring** is a structure $(S, +, 0, \cdot, 1)$ such that
  - ▷ $(S, +, 0)$ is a commutative monoid,
  - ▷ $(S, \cdot, 1)$ is a monoid such that

$$x(y + y') = xy + xy', \qquad (x + x')y = xy + x'y, \qquad 0x = 0 = x0.$$

- A **dioid** is a semiring in which addition is idempotent: $x + x = x$, for every $x \in S$.
  - ▷ The relation defined by

$$x \leqslant y \quad \Longleftrightarrow \quad x + y = y, \quad \text{for } x, y \in S$$

is a partial order on S, with respect to which $+$ and $\cdot$ are monotone, and 0 is minimal.

- A **Kleene algebra** is a dioid K equipped with a **Kleene star** operation $(-)^* : K \to K$ satisfying, for all $x, y, z \in K$
  - ▷ *Unfold axioms*: $1 + xx^* \leqslant x^*$ and $1 + x^*x \leqslant x^*$,
  - ▷ *Induction axioms*: $z + xy \leqslant y \Rightarrow x^*z \leqslant y$ and $z + yx \leqslant y \Rightarrow zx^* \leqslant y$.

Models of Kleene algebras

- The **relation Kleene algebra** on a set X is the structure

$$K(X) \;:=\; (\mathcal{P}(X \times X), \cup, \cdot, \emptyset_X, Id_X, (-)^*).$$

  - The operation $\cdot$ is the relational composition:

$$(a, b) \in R \cdot S \quad \text{iff} \quad (a, c) \in R \text{ and } (c, b) \in S, \text{ for some } c \in X.$$

  - $Id_X = \{(a, a) \mid a \in X\}$ is the identity relation on X.
  - The operation $(-)^*$ is the reflexive transitive closure operation:

$$R^* = \bigcup_{i \in \mathbb{N}} R^i, \text{ with } R^0 = Id_X \text{ and } R^{i+1} = R\,;R^i.$$

- The **path Kleene algebra** on a 1-polygraph X is the structure

$$K(X) \;:=\; (\mathcal{P}(X_1^*), \cup, \odot, \emptyset, \mathbb{1}, (-)^*).$$

  - The composition $\odot$ is defined, for all $\varphi, \psi \in \mathcal{P}(X_1^*)$, by

$$\varphi \odot \psi \;:=\; \{\, u \star_0 v \mid u \in \varphi \wedge v \in \psi \wedge t_0(u) = s_0(v) \,\}.$$

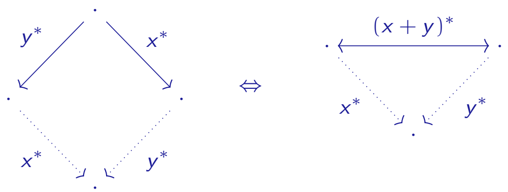
  - $\mathbb{1}$ is the set of all identity arrows of X.
  - The operation $(-)^*$ is defined by $\varphi^* = \bigcup_{i \in \mathbb{N}} \varphi^i$, with $\varphi^0 = \mathbb{1}$ and $\varphi^{i+1} = \varphi \odot \varphi^i$.

Church-Rosser Theorem (algebraic formulation)

Theorem. (Church-Rosser Theorem *à la* Struth, 2002)

For all x, y in a Kleene algebra

$$y^*x^* \leqslant x^*y^* \;\Leftrightarrow (x+y)^* \leqslant x^*y^*.$$



Newman's Theorem (algebraic formulation)

- Algebraic notion of termination (Desharnais-Möller-Struth, 2011).
- A **test** in a diod S is an element $p \leqslant 1$ having a **complement** wrt 1, that is

  there is $q \in S$ such that $p + q = 1$ and $pq = 0 = qp$.
- The set $\mathit{test}(S)$ of all tests of S is a Boolean algebra (complemented distributive lattice)
  - The complement of a test p is unique, and denoted by $\neg p$.
  - Standard Boolean operations:
    - implication: $p \to q = \neg p + q$
    - complementation: $p - q = p \cdot \neg q$

Newman's Theorem (algebraic formulation)

- A semiring S is **modal** if for every x in S there are **forward** and **backward** operators

$$|x\rangle, \langle x| : test(S) \to test(S)$$

satisfying the following axioms:

$$|x\rangle p \leqslant q \quad \Leftrightarrow \quad \neg qxp \leqslant 0 \qquad \text{and} \qquad \langle x|p \leqslant q \quad \Leftrightarrow \quad px\neg q \leqslant 0$$

$$|xy\rangle p = |x\rangle|y\rangle p \qquad \text{and} \qquad \langle xy|p = \langle y|\langle x|p$$

- If x models a **set of transitions** in S, and p represents a **subset of states** on which x acts
  - ▷ $|x\rangle p$ represents the set of all states from which there is a x-transition to p.
  - ▷ $\langle x|p$ represents the set of all states from which there is a x-transition from p.

- Meaning of the first axiom:
  - ▷ If $|x\rangle p \leqslant q$, then it is impossible to make an x-transition from outside q (that is $\neg q$) into p
  - ▷ that is, $(\neg qxp) \equiv ($ part of x that has onlytransitions from $\neg q$ into p $) \equiv \emptyset$.

Newman's Theorem (algebraic formulation)

- A **modal Kleene algebra** is a Kleene algebra that is also a modal semiring.

- For proofs of Newman's like theorems we need of Noetherian induction.

  ▷ An element x in a modal Kleene algebra K is **Noetherian** if 0 is the unique post-fixpoint of $|x\rangle$:

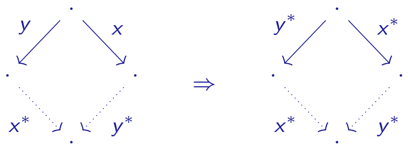
$$p \leqslant |x\rangle p \Rightarrow p \leqslant 0$$

holds for every $p \in test(K)$.

- Newman's Lemma in a modal Kleene Algebra:

Theorem. (Desharnais-Möller-Struth, 2004)
In a modal Kleene algebra K with complete test algebra, if $x + y$ is Noetherian, then

$$\langle y||x\rangle \leqslant |x^*\rangle\langle y^*| \;\Rightarrow\; \langle y^*||x^*\rangle \leqslant |x^*\rangle\langle y^*|.$$



(Co)Domain semirings

- A purely equational approach for modal Kleene algebras (Desharnais-Möller-Struth, 2003).

- A **domain (semiring)** is a semiring $(S, +, \cdot, 0, 1)$ with a **domain operation**

$$d : S \to S$$

satisfying, for all $x, y \in S$,

$$x \leqslant d(x)x, \qquad d(xy) = d(xd(y)), \qquad d(x) \leqslant 1,$$
$$d(0) = 0, \qquad d(x + y) = d(x) + d(y).$$

- A **codomain** is a semiring S with a **codomain operation** $r : S \to S$ such that S^{op} is a domain.

- The **modal diamond operators** are defined, for $x \in S$ and $p \in S_d$, by

$$|x\rangle p = d(xp), \qquad \langle x| p = r(px).$$

  - The **domain algebra** of S is the set of fixpoints of d:

$$S_d := \{x \in S \mid d(x) = x\} = d(S)$$

    - It contains the largest Boolean subalgebra of S bounded by 0 and 1.
    - However, complementation in S_d cannot be expressed.
    - Complementation in domain semirings requires an **antidomain** operator.

Anti(co)domain semirings

- An **antidomain (semiring)** is a semiring $(S, +, \cdot, 0, 1)$ with an **antidomain operation**

$$ad : S \to S$$

such that, for all $x, y \in S$,

$$ad(x)x = 0, \quad ad(xy) \leqslant ad(x\, ad^2(y)), \quad ad^2(x) + ad(x) = 1.$$

  - Setting $d = ad^2$, we recover a domain semiring.
  - The subalgebra S_d is the maximal Boolean subalgebra of $\{x \in S \mid x \leqslant 1\}$.
  - We have $S_d = ad(S)$ and

$$\neg \;:=\; ad_{|S_d}$$

acts as Boolean complementation on S_d.

- An **anticodomain** is a semiring S with an **anticodomain operation** $ar : S \to S$ such that S^{op} is an antidomain.

- A **Boolean modal semiring** S is a antidomain that is also an anticodomain.

  - By maximality, the domain and range algebras coincide: $S_d = S_r$.

- A **Boolean modal Kleene algebra** is a Kleene algebra that is a Boolean modal semiring.

Models of Boolean modal Kleene algebra

- In the relational Kleene algebra $K(X)$ on a set X

$$K(X) := (\mathcal{P}(X \times X), \cup, \cdot, \emptyset_X, Id_X, (-)^*).$$

  - The subidentity relations below Id_X form its greatest Boolean subalgebra between $\emptyset_X$ and Id_X.
    - It is isomorphic to the power set algebra $\mathcal{P}(X)$.
    - Every subalgebra of $K(X)$ is a *relation Kleene algebra*.

- $K(X)$ extends to a Boolean modal Kleene algebra by setting

$$d(R) = \{(a, a) \mid \exists b \in X.\ (a, b) \in R\}, \qquad r(R) = \{(a, a) \mid \exists b.\ (b, a) \in R\}.$$

  - The antidomain and anticodomain operations are given by complementation:

$$ad(R) = Id_X \setminus d(R), \qquad ar(R) = Id_X \setminus r(R).$$

  - Diamond operator:

$$|R\rangle P = \{(a, a) \mid \exists b \in X.\ (a, b) \in R \wedge (b, b) \in P\}.$$

Models of Boolean modal Kleene algebra

- The path Kleene algebra on a 1-polygraph X

$$K(X) := (\mathcal{P}(X_1^*), \cup, \odot, \emptyset, \mathbb{1}, (-)^*).$$

extends to a Boolean modal Kleene algebra by setting

$$d(\varphi) = \{1_{s(u)} \mid u \in \varphi\} \qquad r(\varphi) = \{1_{t(u)} \mid u \in \varphi\}$$

where 1_x denotes the identity arrow on the object $x \in X_0$.

  - Antidomain and anticodomain maps are defined by complementation

$$ad(\varphi) = \mathbb{1} \setminus d(\varphi), \qquad ar(\varphi) = \mathbb{1} \setminus r(\varphi).$$

  - Forward diamond operator:

$$|\varphi\rangle p = \{1_{s(u)} \mid u \in \varphi \wedge t(u) \in p\},$$

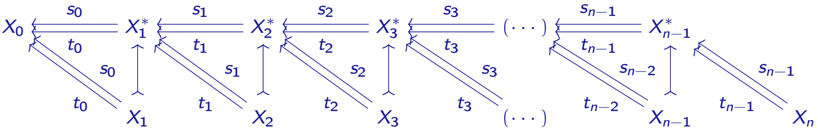
where $p \subseteq \mathbb{1}$ is some set of identity arrows.

- Reachability along a relation in the relation model is replaced by reachability along a set of paths in the path model.

Part II:
Abstract coherence by rewriting

Polygraphs

▶ Consider an n-polygraph $X = (X_0, X_1, \ldots, X_n)$



▷ It induces an abstract rewriting system on the free $(n-1)$-category X_{n-1}^*.

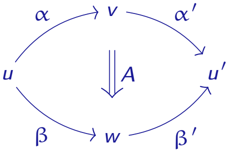
▷ We extend the (abstract) rewriting properties on X:

termination / **confluence** / **locally confluence** / **convergence**.

Squier's completion

- Let X be a convergent n-polygraph.

- A **family of generating confluences** of X is a cellular extension of the $(n, n-1)$-category $X_n^{\top}$ that contains exactly one $(n+1)$-cell

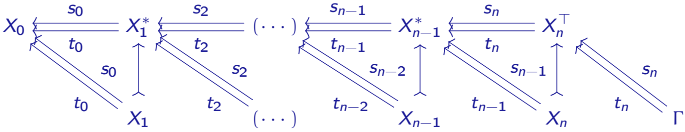


for every critical branching (α, β) of X.

- A **Squier's completion** of the n-polygraph X is the $(n+1, n-1)$-polygraph

$$\mathcal{S}(X) = (X, \Gamma)$$

where Γ is a chosen family of generating confluences of X.

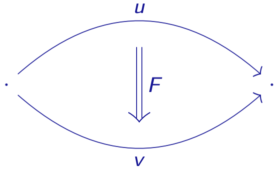


Squier's completion and finite derivation type

Theorem.

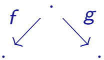
If X is a convergent n-polygraph, then the $(n+1, n-1)$-polygraph $\mathcal{S}(X) = (X, \Gamma)$ is acyclic, that is the $(n, n-1)$-category $X_n^\top/\Gamma$ is aspherical:

for any n-cells $u||v$ in the free $(n, n-1)$-category $X_n^\top$, there is an $(n+1)$-cell $F : u \Rrightarrow v$ in $X_n^\top(\Gamma)$ such that $s_n(F) = u$ and $t_n(F) = v$.

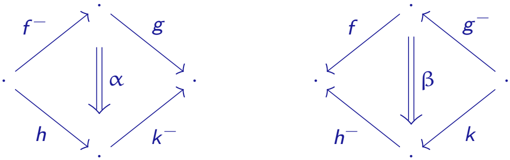


- The proof relies on the following two coherent confluent results:
  - Coherent Newman's lemma.
  - Coherent Church-Rosser theorem.

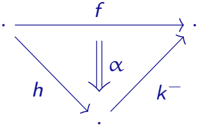
Coherent confluence

- Let X be an n-polygraph. A cellular extension Γ of $X_n^\top$ is a
  - **confluence filler of a branching** 

 of X if there exist n-cells h, k in X_n^*, and $(n+1)$-cells α, β in $X_n^\top[\Gamma]$ with shapes:



  - **confluence filler** of an n-cell f in $X_n^\top$ if there exist n-cells h, k in X_n^* and an $(n+1)$-cell α in $X_n^\top[\Gamma]$ of the shape:



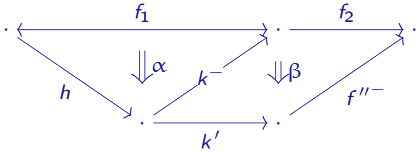
  - **confluence filler** of X if Γ is a confluence filler for each of its branchings.
  - **Church-Rosser filler** of X when it is a confluence filler of every n-cell in $X_n^\top$.

Coherent confluence

Theorem. (Coherent Church-Rosser filler lemma)

Let X be an n-polygraph, and Γ a cellular extension of $X_n^\top$.
Then Γ is a confluence filler for X if and only if Γ is a Church-Rosser filler for X.

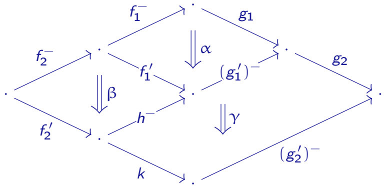
Proof.



Theorem. (Coherent Newman filler lemma)

Let X be a terminating n-polygraph, and Γ a cellular extension of $X_n^\top$.
Then Γ is a local confluence filler if and only if Γ is a confluence filler for X.

Proof.



Part III:
Calculating coherent proofs in higher modal Kleene algebras

Higher dioids

- A 0-**dioid** is a bounded distributive lattice:
  - *i.e.*, an idempotent semiring $(S, +, 0, \cdot, 1)$ whose multiplication $\cdot$ is commutative and idempotent, and $x \leqslant 1$, for every $x \in S$.

- For $n \geqslant 1$, an n-**diod** is a structure $(S, +, 0, \odot_i, 1_i)_{0 \leqslant i < n}$ such that
  - $(S, +, 0, \odot_i, 1_i)$ is a dioid for $0 \leqslant i < n$,
  - The **lax interchange laws** hold, for all $0 \leqslant i < j < n$,

$$(x \odot_j x') \odot_i (y \odot_j y') \leqslant (x \odot_i y) \odot_j (x' \odot_i y'),$$

  - Higher units are idempotents of lower multiplications, for all $0 \leqslant i < j < n$,

$$1_j \odot_i 1_j = 1_j.$$

Remark. (why a 0-dioid is a bounded distributive lattice)

- Consider the path Kleene algebra $K(X) = (\mathcal{P}(X_1^*), \cup, \odot, \emptyset, \mathbb{1}, (-)^*)$ on a 1-polygraph X with domain $d(\varphi) = \{1_{s(u)} \mid u \in \varphi\}$.
- The domain algebra $K(X)_d$ is isomorphic to the power set $\mathcal{P}(X_0)$.
  - It forms a bounded distributive lattice with $+$ as join, $\cdot$ as meet, $\emptyset$ as bottom and $\mathbb{1}$ as top.
- The idempotence and commutativity of the multiplication operation simulate the properties of a set of identity 1-cells.

Higher modal semirings

- An **antidomain 0-semiring** is a 0-diod.
- For $n \geqslant 1$, an **antidomain n-semiring** is a n-dioid $(S, +, 0, \odot_i, 1_i)_{0 \leqslant i < n}$ equipped with antidomain maps $(ad_i : S \to S)_{0 \leqslant i < n}$ such that
  - $(S, +, 0, \odot_i, 1_i, ad_i)$ is an antidomain semiring, for all $x, y \in S$,
$$ad_i(x)x = 0, \quad ad_i(xy) \leqslant ad_i(x\, ad^2(y)), \quad ad_i^2(x) + ad_i(x) = 1.$$
  - $ad_{i+1} \circ ad_i = ad_i$.
- An **anticodomain n-semiring** is a n-dioid S such that $S^{op} = (S_i^{op})_{0 \leqslant i < n}$ is a antidomain n-semiring. The codomain operators are denoted by $(ar_i : S \to S)_{0 \leqslant i < n}$.
- A **Boolean modal n-semiring** is an antidomain n-semiring that is also an anticodomain n-semiring for $n \geqslant 1$, and a Boolean algebra for $n = 0$.

Properties.

- Setting $d_i = ad_i^2$ and $r_i = ar_i^2$, we recover a domain and codomain n-semirings.
- The ***i*-dimensional domain algebra** is the set of fixpoints $S_i := d_i(S) = ad_i(S)$.
  - We have $S_0 \subseteq S_1 \subseteq S_2 \subseteq \ldots \subseteq S_{n-1} \subseteq S$.
  - The subalgebra $(S_i, +, 0, \odot_i, 1_i, ad_i)$ is a Boolean algebra, and $\neg_i := ad_{|S_i}$ acts as Boolean complementation on S_i.

Modal n-Kleene algebra

- An **n-Kleene algebra** is an n-dioid K equipped with operations $(-)^{*_i} : K \to K$ such that
  - $(K, +, 0, \odot_i, 1_i, (-)^{*_i})$ is a Kleene algebra for $0 \leqslant i < n$,
  - For $0 \leqslant i < j < n$, the operation $(-)^{*_j}$ is a lax morphism wrt i-whiskering of j-dimensional elements:

$$\varphi \odot_i A^{*_j} \leqslant (\varphi \odot_i A)^{*_j} \qquad A^{*_j} \odot_i \varphi \leqslant (A \odot_i \varphi)^{*_j}$$

for all $A \in K$, $\varphi \in K_j$.

- A **modal n-Kleene algebra** is a n-Kleene algebra that is a modal n-semiring (domain and codomain semiring).

- A **Boolean modal n-Kleene algebra** is a n-Kleene algebra that is a Boolean modal n-semiring.

- The **forward** and **backward i-diamond** operators in a modal n-semiring are, for all $0 \leqslant i < n$, $A \in S$ and $\varphi \in S_i$,

$$|A\rangle_i(\varphi) := d_i(A \odot_i \varphi), \qquad \langle A|_i(\varphi) := r_i(\varphi \odot_i A).$$

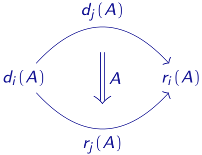
Globular Kleene algebras

- A modal n-Kleene algebra K is **globular** if the following **globular relations** hold for $0 \leqslant i < j < n$ and $A, B \in K$:

$$d_i \circ d_j = d_i, \qquad d_i \circ r_j = d_i, \qquad r_i \circ d_j = r_i, \qquad r_i \circ r_j = r_i,$$

$$d_j(A \odot_i B) = d_j(A) \odot_i d_j(B), \qquad r_j(A \odot_i B) = r_j(A) \odot_i r_j(B).$$

  - An element A in K is a collection of cells, and for $i < j$:



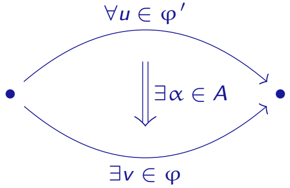
  - $d_k(A)$ is the set of k-cells that are k-sources of some cells belonging to A.
  - $r_k(A)$ is the set of k-cells that are k-targets of some cells belonging to A.
  - We have

$$A \odot_i B = (A \odot_i r_i(A)) \odot_i (d_i(B) \odot_i B) = (A \odot_i d_i(B)) \odot_i (r_i(A) \odot_i B).$$

Confluence fillers

- Let K be a globular n-modal Kleene algebra and $0 \leqslant i < j < n$.
- Given $A \in K$ and $\varphi, \varphi' \in K_j$, we have
$$|A\rangle_j(\varphi) \geqslant \varphi' \quad \text{iff} \quad d_j(A \odot_j \varphi) \geqslant \varphi'.$$
- In the polygraphic model:
$$\forall u \in \varphi', \exists v \in \varphi \text{ and } \exists \alpha \in A \text{ such that } s_j(\alpha) = u \text{ and } t_j(\alpha) = v.$$

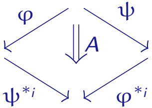
$$|A\rangle_j(\varphi) \geqslant \varphi'$$



Confluence fillers

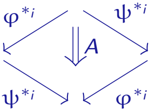
- Let φ, ψ in K_j. An element A in K is a
  - **local i-confluence filler** for (φ, ψ) if

$$|A\rangle_j(\psi^{*_i} \odot_i \varphi^{*_i}) \geqslant \varphi \odot_i \psi$$



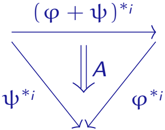
  - **i-confluence filler** for (φ, ψ) if

$$|A\rangle_j(\psi^{*_i} \odot_i \varphi^{*_i}) \geqslant \varphi^{*_i} \odot_i \psi^{*_i}$$



  - **i-Church-Rosser filler** for (φ, ψ) if

$$|A\rangle_j(\psi^{*_i} \odot_i \varphi^{*_i}) \geqslant (\psi + \varphi)^{*_i}$$



- Note that $(\psi + \varphi)^{*_i} \geqslant \varphi^{*_i} \odot_i \psi^{*_i} \geqslant \varphi \odot_i \psi$.

Completion fillers

- The **right** and **left** i**-whiskering** of $A \in K$ by $\varphi \in K_j$ is

$$A \odot_i \varphi \qquad \text{and} \qquad \varphi \odot_i A$$

- In the proofs, we use **completion of an** i**-confluence filler** A of a pair (φ, ψ) in K_j:
  - The j**-dimensional** i**-whiskering of** A

$$(\varphi + \psi)^{*_i} \odot_i A \odot_i (\varphi + \psi)^{*_i} \in K$$

  - The i**-whiskered** j**-completion of** A, denoted by $\hat{A}^{*_j}$, is

$$\big((\varphi + \psi)^{*_i} \odot_i A \odot_i (\varphi + \psi)^{*_i}\big)^{*_j} \in K$$

- The completion $\hat{A}$ of a confluence filler A absorbs whiskers:

for every $\xi \leqslant (\varphi + \psi)^{*_i}$, $\qquad \xi \odot_i \hat{A}^{*_j} \leqslant \hat{A}^{*_j} \qquad$ and $\qquad \hat{A}^{*_j} \odot_i \xi \leqslant \hat{A}^{*_j}$.

Coherent Church-Rosser and Newman in globular MKA

Theorem. (Calk-Goubault-M.-Struth, 2023)

Let K be a globular modal n-Kleene algebra and $0 \leqslant i < j < n$. Given $\varphi, \psi \in K_j$ and an i-confluence filler $A \in K$ of (φ, ψ), we have

$$|\hat{A}^{*_j}\rangle_j(\psi^{*_i} \odot_i \varphi^{*_i}) \geqslant (\varphi + \psi)^{*_i},$$

where $\hat{A}$ is the j-dimensional i-whiskering of A, and thus $\hat{A}^{*_j}$ is an i-Church-Rosser filler for (φ, ψ).

Theorem. (Calk-Goubault-M.-Struth, 2023)

Let K be a Boolean globular modal n-Kleene algebra, and $0 \leqslant i < j < n$, such that

- ▷ $(K_i, +, 0, \odot_i, 1_i, \neg_i)$ *is a complete Boolean algebra,*
- ▷ K_j *is i-continuous.*

Let $\psi \in K_j$ be i-Noetherian and $\varphi \in K_j$ i-well-founded.
If A is a local i-confluence filler for (φ, ψ), then

$$|\hat{A}^{*_j}\rangle_j(\psi^{*_i} \odot_i \varphi^{*_i}) \geqslant \varphi^{*_i} \odot_i \psi^{*_i},$$

that is $\hat{A}^{*_j}$ is a confluence filler for (φ, ψ).

Polygraphic model of higher Kleene algebras

- Let (X, Γ) be an $(n+1, n-1)$-polygraph.
- The $(n+1)$-modal Kleene algebra $K(X, \Gamma)$ is the full $(n+1)$-path algebra:

$$K(X) := \mathcal{P}\big(X^*_{n-1}(X_n)[\Gamma]\big),$$

  - With multiplication

$$A \odot_i B := \{\alpha \star_i \beta \mid \alpha \in A \wedge \ \beta \in B \wedge t_i(\alpha) = s_i(\beta)\}$$

for all $A, B \in K(X)$.

  - The unit for $\odot_i$ is the set

$$\mathbb{1}_i \ = \ \{\iota_i^{n+1}(u) \mid u \in X^*_{n-1}(X_n)[\Gamma]_i\}.$$

  - Addition is the set union $\cup$, and the ordering is the set inclusion.
  - i-domain and i-codomain maps:

$$d_i(A) := \{\iota_i^{n+1}(s_i(\alpha)) \mid \alpha \in A\}, \qquad r_i(A) := \{\iota_i^{n+1}(t_i(\alpha)) \mid \alpha \in A\}.$$

  - i-antidomain and i-anticodomain maps:

$$ad_i(A) := \mathbb{1}_i \setminus \{\iota_i^{n+1}(s_i(\alpha)) \mid \alpha \in A\}, \qquad ar_i(A) := \mathbb{1}_i \setminus \{\iota_i^{n+1}(t_i(\alpha)) \mid \alpha \in A\}.$$

  - The i-star is $A^{*_i} = \bigcup_{k\in\mathbb{N}} A^{k_i}$, with $A^{0_i} := \mathbb{1}_i$ and $A^{k_i} := A \odot_i A^{(k-1)_i}$.

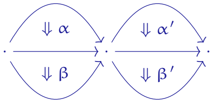
Polygraphic model of higher Kleene algebras

Proposition.

Let (X, Γ) *be an* $(n+1, n-1)$*-polygraph. Then* $K(X, \Gamma)$ *is a Boolean globular modal* $(n+1)$*-Kleene algebra with converse.*

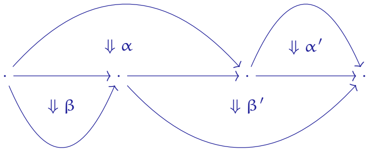
- Lax exchange law: for $A, A', B, B' \in K(P, \Gamma)$ and $0 \leqslant i < j < n+1$

$$(A \odot_j B) \odot_i (A' \odot_j B') \subseteq (A \odot_i A') \odot_j (B \odot_i B').$$



$$(\alpha \star_j \beta) \star_i (\alpha' \star_j \beta') = (\alpha \star_i \alpha') \star_j (\beta \star_i \beta') \in (A \odot_i A') \odot_j (B \odot_i B')$$

  - The lax exchange law is not reduced to an equality:



Polygraphic model of higher Kleene algebras

▶ Consequences of coherent Church-Rosser and Newman results in globular MKA, in the polygraphic model:

Theorem. (Coherent Church-Rosser filler lemma)

Let X be an n-polygraph, and Γ a cellular extension of $X_n^\top$.
Then Γ is a confluence filler for X if and only if Γ is a Church-Rosser filler for X.

Theorem. (Coherent Newman filler lemma)

Let X be a terminating n-polygraph, and Γ a cellular extension of $X_n^\top$.
Then Γ is a local confluence filler if and only if Γ is a confluence filler for X.

Conclusion:
Work in progress

Work in progress

Problem A. Calculating (finite) polygraphic resolutions from (finite) rewriting systems.

- ▷ Algebraic formulation of normalisation strategies in modal Kleene algebras.
- ▷ In low dimension, Squier's theorem for ARS (Calk-Goubault-M., 2021).
- ▷ Higher normalisation strategies in ω-quantales (M.-Struth).

Problem B. Polygraphic resolutions for algebraic structures expressible by polygraphs.

- ▷ Formalisation of the coherent critical branching lemma (strings, terms, terms modulo).
- ▷ (Algebraically enriched) n-Kleene algebras.

Problem C. Relationship between polygraphic resolution and directed homotopy.

- ▷ Concurrent Kleene algebras are modal 2-Kleene algebras
  (with $1_0 = 1_1$ and commutativity of $\odot_1$).
  - ▷ A concurrent Kleene algebra offers choice, iteration, and two composition operators for sequential and concurrent execution.
  - ▷ Acyclicity in concurrent Kleene algebras?

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