The rabbit calculus:
convolution products on double categories
and categorification of rule algebra

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The quest for causality in rewriting theory

An important insight coming from Huet and Lévy:

In order to track the causality structure relating different $\beta$-redexes, one needs to consider rewriting paths modulo permutations of the form

$$\begin{align*}
(\lambda x. (\lambda y. x)) M Q \\
(\lambda y. M) Q \\
(\lambda x. (\lambda y. x)) M N \\
(\lambda y. M) N \\
\end{align*}$$
The quest for causality in rewriting theory

In the \( \lambda \)-calculus and term rewriting systems

A tradition based on optimality and residual theory

- the notion of Lévy families in the \( \lambda \)-calculus (Lévy 1980)
- their generalisation to any CRS (Asperti, Laneve 1995)
- a residual theory based on the notion of trek (PAM, 2002)

More recently, in categorical graph rewriting

- the notion of tracelet emerging in the work by Nicolas Behr.

Our ambition in this work is to initiate a convergence between these lines by revisiting/categorifying the work on tracelets using double categories.
Double categories

Definition. A (weak) **double category** $\mathcal{D}$ consists of

- a category $\mathcal{D}_0$ of objects,
- a category $\mathcal{D}_1$ of horizontal maps,
- a pair of **source** and **target** functors

$$
\mathcal{D}_0 \xleftarrow{T} \mathcal{D}_1 \xrightarrow{S} \mathcal{D}_0
$$

- a **horizontal composition** functor

$$
\Diamond_h : \mathcal{D}_1 \times \mathcal{D}_0 \mathcal{D}_1 \rightarrow \mathcal{D}_1
$$

- a **horizontal identity** functor

$$
idh : \mathcal{D}_0 \rightarrow \mathcal{D}_1
$$

satisfying a number of **associativity** and **neutrality** properties.
The category $\mathbb{D}_0$ of vertical maps

A morphism in the category $\mathbb{D}_0$ is represented as a vertical map

\[
\begin{array}{c}
A \\
\downarrow a \\
A'
\end{array}
\]

which may be composed vertically with other vertical maps.
The category $\mathbb{D}_1$ of horizontal maps

An object in the category $\mathbb{D}_1$ is represented as a horizontal map

$$B \xleftarrow{r} A$$

A morphism in the category $\mathbb{D}_1$ is represented as a double cell

$$B \xleftarrow{r} A$$

$$\downarrow b \quad \downarrow \alpha \quad \downarrow a$$

$$\downarrow$$

$$B' \xleftarrow{r'} A'$$

which may be composed vertically with other double cells.
The category $\mathbf{D}_1$ of horizontal maps

We often find convenient to use the pictorial notation

for the double cell usually noted

\[
\begin{array}{c}
\begin{array}{c}
B \quad r \quad A \\
B' \quad r' \quad A'
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
b \\ \alpha \\ a
\end{array}
\end{array}
\]
The category $\mathcal{D}_2$ of paths of length 2

Every double category $\mathcal{D}$ comes with

a category $\mathcal{D}_2 = \mathcal{D}_1 \times_{\mathcal{D}_0} \mathcal{D}_1$ of horizontal paths of length 2
defined as the limit of the diagram of functors

in the category $\textbf{Cat}$ of categories and functors.
The category $D_2$ of paths of length 2

A typical morphism of $D_2$ has the shape

$$
\begin{array}{cccc}
C & \leftarrow & s & \rightarrow & B & \leftarrow & r & \rightarrow & A \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
C' & \leftarrow & s' & \rightarrow & B' & \leftarrow & r' & \rightarrow & A' \\
\end{array}
$$

which we also like to depict as

$$
\begin{array}{cccc}
s & \beta & b & \alpha & a \\
| & | & | & | & | \\
s' & \beta' & b' & \alpha' & a'
\end{array}
$$
The category $D_3$ of paths of length 3

Every double category $D$ comes with

a category $D_3$ of horizontal paths of length 3 defined as the limit of the diagram of functors

in the category $\text{Cat}$ of categories and functors.
The category $D_3$ of paths of length 3

A typical morphism of $D_3$ has the shape

which we also like to depict as
The category $\mathcal{D}_4$ of paths of length 4

Every double category $\mathcal{D}$ comes with

a category $\mathcal{D}_4$ of horizontal paths of length 4

defined as the limit of the diagram of functors

in the category $\mathbf{Cat}$ of categories and functors.
The category $\mathbb{D}_4$ of paths of length 4

A typical morphism of $\mathbb{D}_4$ has the shape

\[
\begin{array}{cccccc}
E & \xleftarrow{u} & D & \xleftarrow{t} & C & \xleftarrow{s} & B & \xleftarrow{r} & A \\
& \downarrow{e} & \downarrow{\delta} & \downarrow{d} & \downarrow{\gamma} & \downarrow{c} & \downarrow{\beta} & \downarrow{b} & \downarrow{\alpha} & \downarrow{a} \\
E' & \xleftarrow{u'} & D' & \xleftarrow{t'} & C' & \xleftarrow{s'} & B' & \xleftarrow{r'} & A'
\end{array}
\]

which we also like to depict as

[Diagram of the category $\mathbb{D}_4$ with labeled arrows and nodes]
Unbiased presentation of a double category

Every double category $D$ comes equipped with a family of functors

$$h_n : D_n \longrightarrow D_1$$

called the **horizontal composition** functors, and satisfying a number of **associativity** and **neutrality** properties.

This leads to an alternative (unbiased) definition of (weak) double category.

Note that the functors $h_2$ and $h_0$ coincide with the functors $\diamond_h$ and $idh$

$$h_2 = \diamond_h : D_2 \longrightarrow D_1$$

$$h_0 = idh : D_0 \longrightarrow D_1$$
The double category **DPO** of double pushouts

The double category $\mathbf{D} = \mathbf{DPO}$ on an adhesive category $\mathbf{C}$

- whose objects are objects $A, B, C$ of the cohesive category $\mathbf{C}$,
- whose horizontal maps $M = (S, s, t)$ are spans in $\mathbf{C}$,
- whose vertical maps $\lambda_A : A \to A'$ are monos in $\mathbf{C}$,
- whose double cells $\theta : M \Rightarrow M'$ are monos $\lambda_\theta : S \to S'$ making the pushout diagram commute:

$$
\begin{array}{c}
B & \xleftarrow{M} & A \\
\downarrow{\lambda_B} & \downarrow{\theta} & \downarrow{\lambda_A} \\
B' \ & \xleftarrow{M'} & A'
\end{array}
\quad
\begin{array}{c}
B & \xleftarrow{t} & S & \xrightarrow{s} & A \\
\downarrow{\lambda_B} & \downarrow{\lambda_\theta} \downarrow{PO} & \downarrow{PO} & \downarrow{\lambda_A} \\
B' \ & \xleftarrow{t'} & S' \ & \xrightarrow{s'} & A'
\end{array}
$$
Rewriting rules as covariant presheaves

A rewriting rule provided by a horizontal map

\[ r : B \leftarrow A \]

is described in our framework as the representable presheaf

\[ \hat{\Delta}_r : \mathbb{D}_1 \rightarrow \text{Set} \]

which associates to every horizontal map

\[ u : B' \leftarrow A' \]

the set

\[ \mathbb{D}_1(r, u) \]

of all possible implementations of the transformation \( u \) by the rule \( r \).
Category of elements of a presheaf

The Grothendieck construction
Elements of a covariant presheaf

Recall that an element

$$(a, x) \in \text{Elts}(F)$$

of a covariant presheaf

$$F : C \longrightarrow \text{Set}$$

is defined as a pair

$$\left( a \in C \ , \ x \in F(a) \right)$$

consisting of

- an object $a$ of the underlying category $\mathbf{C}$,
- an element $x$ of the set $F(a)$. 

---
Elements of a covariant presheaf

We find enlightening to draw such a pair

\[ \left( a \in \mathcal{C}, \quad x \in F(a) \right) \in \text{Elts}(F) \]

in the following way

\[ \begin{array}{c}
F \\
\downarrow \\
x \\
\downarrow \\
a
\end{array} \]

with the intuition that the element

\[ x \in F(a) \]

provides a **witness** of the covariant presheaf \( F \) at instance \( a \in \mathcal{C} \).
Covariant action of a presheaf

By definition of a covariant presheaf

\[ F : \mathcal{C} \rightarrow \text{Set} \]

every element

\[ \left( a \in \mathcal{C}, x \in F(a) \right) \in \text{Elts}(F) \]

and morphism of the category \( \mathcal{C} \)

\[ \gamma : a \rightarrow a' \]

induces an element

\[ \left( a' \in \mathcal{C}, \gamma \cdot x = F(\gamma)(x) \in F(a') \right) \in \text{Elts}(F) \]
Covariant action of a presheaf

This means that every diagram

\[
\begin{array}{c}
\text{a} \\
\downarrow \gamma \\
\text{a}'
\end{array}
\xrightarrow{F}
\begin{array}{c}
\text{x} \\
\downarrow \gamma \\
\text{a}'
\end{array}
\]

can be completed into the diagram

\[
\begin{array}{c}
\text{a} \\
\downarrow \gamma \\
\text{a}'
\end{array}
\xrightarrow{F}
\begin{array}{c}
\text{x} \\
\downarrow \gamma \\
\text{a}'
\end{array}
\]

where \( \gamma \cdot x = F(\gamma)(x) \).
The category of elements

The category \( \text{Elts}(F) \) of elements of a covariant presheaf

\[
F : \mathcal{C} \longrightarrow \text{Set}
\]

is defined in the following way:

- its objects are the elements \((a, x)\) of the covariant presheaf \( F \)
- its morphisms

\[
(f, x) : (a, x) \longrightarrow (a', x')
\]

are the pairs consisting of a morphism

\[
f : a \longrightarrow a'
\]

of the category \( \mathcal{C} \) and an element \( x \in F(a) \) such that

\[
f \cdot x = F(f)(x) = x'
\]
The category of elements

The category of elements

\[ \text{Elts}(F) \]

associated to a covariant presheaf

\[ F : C \rightarrow \text{Set} \]

comes equipped with a projection functor

\[ \pi_F : \text{Elts}(F) \rightarrow C \]

which transports every element

\( (a, x) \in \text{Elts}(F) \)

to the object \( a \in C \) of the underlying category \( C \).

**Fact.** The functor \( \pi_F \) defines a **discrete opfibration**.
Grothendieck opfibrations

**Definition.** A functor

\[ p : E \longrightarrow C \]

is an opfibration when there exists an opcartesian morphism

\[
\begin{array}{ccc}
R & \xrightarrow{f} & S \\
\downarrow & & \downarrow \\
A & \xrightarrow{u} & B
\end{array}
\]

for every object \( R \in p^{-1}(A) \) and every morphism \( u : A \rightarrow B \).
Opcartesian morphisms

A morphism $f : R \to S$ in $E$ is opcartesian above $u : A \to B$ in $C$ when the following property holds:

for every map $g : R \to T$

for every map $v : B \to C$

such that $p(g) = v \circ u$

there exists

a unique map $h : S \to T$

such that $h \circ f = g$

and $p(h) = v$. 
The Grothendieck correspondence

The projection functor

\[ \pi_F : \text{Elts} (F) \longrightarrow C \]

is a discrete opfibration. Indeed, every diagram

\[
\begin{array}{ccc}
  x & \xrightarrow{f} & a' \\
  \downarrow \pi_F & & \downarrow \\
  a & \xrightarrow{f} & a'
\end{array}
\]

can be completed with the opcartesian morphism \((f, x)\) as follows:

\[
\begin{array}{ccc}
  x & \xrightarrow{(f, x) \in \text{Elts} (F)} & f \cdot x \\
  \downarrow \pi_F & & \downarrow \\
  a & \xrightarrow{f \in C} & a'
\end{array}
\]
The Grothendieck correspondence

Moreover, every natural transformation

\[ \theta : C \rightarrow \text{Set} \]

induces a commutative diagram of discrete opfibrations:

\[ \begin{array}{ccc}
\text{Elts}(F) & \xrightarrow{\text{Elts}(\theta)} & \text{Elts}(G) \\
\pi_F & & \pi_G \\
C & \xleftarrow{\theta} & C
\end{array} \]
The Grothendieck correspondence

**Fact.** This induces a categorical equivalence between

- The category \([C, \text{Ens}]\) of **covariant presheaves**

\[
F, G : C \longrightarrow \text{Set}
\]

and natural transformations between them.

- The slice category \(\text{DiscOpFib}/C\) of **discrete opfibrations** above \(C\).

Moreover, there is an adjunction

\[
\begin{array}{ccc}
\text{Cat}/C & \Downarrow & \text{DiscOpFib}/C \\
\rotatebox{90}{Free} & \circlearrowright & \\
\rotatebox{90}{Inclusion}
\end{array}
\]
The Day convolution product

A construction on monoidal categories
The Day convolution product

Given two covariant presheaves

\[ F, G : C \longrightarrow \text{Set} \]

on a monoidal category \( C \) with tensor product

\[ \otimes : C \times C \longrightarrow C \]

the **Day convolution product** of \( F \) and \( G \) is the covariant presheaf

\[ G \hat{\otimes} F : C \longrightarrow \text{Set} \]

defined by the coend formula

\[
G \hat{\otimes} F = c \mapsto \int_{(b, a) \in C \times C} \mathbb{C}(b \otimes a, c) \times G(b) \times F(a)
\]
The Day convolution product

Equivalently, the convolution product

\[ G \hat{\otimes} F : C \rightarrow \text{Set} \]

may be defined as the **left Kan extension** of the functor

\[ C \times C \xrightarrow{G \times F} \text{Set} \times \text{Set} \xrightarrow{\times} \text{Set} \]

along the tensor product functor:

\[ C \times C \xrightarrow{G \hat{\otimes} F} \text{Set} \times \text{Set} \xrightarrow{\times} \text{Set} \]
What does the coend formula mean?

An element of the coend

$$G \hat{\otimes} F (c) = \int_{(b,a) \in C \times C} C(b \otimes a, c) \times G(b) \times F(a)$$

consists of a morphism

$$b \otimes a \xrightarrow{\gamma} c$$

together with a pair of elements

$$y \in G(b) \quad x \in F(a)$$

considered modulo an equivalence relation $\sim$. 
What does the coend formula mean?

As we did before, we find enlightening to draw the two elements

\[ y \in G(b) \quad x \in F(a) \]

in the following way:

```
\begin{array}{ccc}
  G & \quad & F \\
  \downarrow & & \downarrow \\
  y & \quad & x \\
  \downarrow & & \downarrow \\
  b & \quad & a
\end{array}
```
What does the coend formula mean?

Accordingly, we like to draw the triple

\[ (b \otimes a \xrightarrow{\gamma} c, \quad x \in F(a), \quad y \in G(b) ) \]

in the following way:
What does the coend formula mean?

Suppose given a pair of elements

\[ x \in F(a) \quad y \in G(b) \]

a pair of morphisms

\[ \alpha : a \to a' \quad \beta : b \to b' \]

and a morphism

\[ \gamma : a' \otimes b' \to c \]
What does the coend formula mean?

The situation may be depicted as follows:
What does the coend formula mean?

The diagram may be completed as follows:
What does the coend formula mean?

This equivalence relation $\sim$ defined by the coend

$$G \hat{\otimes} F (c) = \int_{(b,a) \in C \times C} C(b \otimes a, c) \times G(b) \times F(a)$$

identifies every triple of the form

$$\left( b \otimes a \xrightarrow{\beta \otimes \alpha} b' \otimes a' \xrightarrow{\gamma} c, \quad x \in F(a), \quad y \in G(b) \right)$$

with the corresponding triple

$$\left( b' \otimes a' \xrightarrow{\gamma} c, \quad \alpha \cdot x \in F(a'), \quad \beta \cdot y \in G(b') \right)$$
What does the coend formula mean?

Diagrammatically, the equivalence relation $\sim$ identifies the two triples:
The Day convolution product

**Theorem [Day 1970]** The convolution product

$$G, F \mapsto \hat{G} \hat{\otimes} F$$

on a monoidal category \( C \) with tensor product \( \otimes \) defines a functor

$$\hat{\otimes} : [C, Set] \times [C, Set] \to [C, Set]$$

which equips the category of covariant presheaves

\([C, Set]\)

with the structure of a monoidal closed category.

In particular, the convolution product is associative:

$$H \hat{\otimes} (G \hat{\otimes} F) \cong (H \hat{\otimes} G) \hat{\otimes} F$$
A key observation

Fact. The projection functor

\[ \pi_{G \hat{\otimes} F} : \text{Elts}(G \hat{\otimes} F) \to C \]

associated to the Day convolution product

\[ G \hat{\otimes} F : C \to \text{Set} \]

is the free discrete opfibration associated to the functor

\[ \text{Elts}(G) \times \text{Elts}(F) \xrightarrow{\pi_G \times \pi_F} C \times C \xrightarrow{\otimes} C \]

obtained by tensoring the two projection functors

\[ \text{Elts}(G) \xrightarrow{\pi_G} C \quad \text{Elts}(F) \xrightarrow{\pi_F} C \]
Construction of the free discrete opfibration

Step 0. We start from the functor

\[
\text{Elts}(G) \times \text{Elts}(F) \xrightarrow{\pi_G \times \pi_F} C \times C \xrightarrow{\otimes} C
\]

whose objects in the source category are pairs

\[
\left( x \in F(a) , \ y \in G(b) \right)
\]

may be depicted in the following way:

\[
\begin{array}{ccc}
G & \xrightarrow{y} & F \\
\downarrow{b \otimes a} & & \\
\end{array}
\]

\[
\begin{array}{ccc}
x \\
\end{array}
\]
Construction of the free discrete opfibration

**Step 1.** We replace the functor by its free split opfibration

\[
\begin{align*}
\text{Elts} (G, F) \xrightarrow{\pi_{G,F}} C
\end{align*}
\]

where the source category \textbf{Elts} \((G, F)\) has objects defined as triples

\[
\left( b \otimes a \xrightarrow{\gamma} c , \quad x \in F(a) , \quad y \in G(b) \right)
\]

which may be depicted in the following way:
Construction of the free discrete opfibration

**Step 1.** We replace the functor by its **free split opfibration**

\[ \text{Elts}(G,F) \xrightarrow{\pi_{G,F}} C \]

whose morphisms in each fiber above \( c \in C \) are of the form:

\[
\begin{array}{c}
G \\
\downarrow \beta \otimes \alpha \\
F \\
\downarrow \gamma \\
c
\end{array}
\quad \xrightarrow{\beta \cdot y \otimes \alpha \cdot x} 
\begin{array}{c}
G \\
\downarrow \beta \cdot y \\
F \\
\downarrow \gamma \\
c
\end{array}
\]
Construction of the free discrete opfibration

Step 2. Replace each fiber category of the opfibration

\[ \text{Elts}(G,F) \xrightarrow{\pi_{G,F}} C \]

by its set of **connected components**, using the equivalence relation:

\[
\begin{align*}
G & \xrightarrow{y} b \otimes a \\
F & \xrightarrow{x} \beta \otimes \alpha \\
& \downarrow \beta \otimes \alpha \\
b' \otimes a' & \xrightarrow{\gamma} c
\end{align*}
\]

\[
\begin{align*}
G & \xrightarrow{\beta \cdot y} b' \otimes a' \\
& \downarrow \gamma \\
c & \xrightarrow{\alpha \cdot x}
\end{align*}
\]
A key observation

From this follows that there exists a cofinal functor

\[ \text{Elts}(G) \times \text{Elts}(F) \longrightarrow \text{Elts}(G \hat{\otimes} F) \]

making the diagram commute:

\[
\begin{array}{ccc}
\text{Elts}(G) \times \text{Elts}(F) & \xrightarrow{\text{cofinal}} & \text{Elts}(G \hat{\otimes} F) \\
\otimes \circ (\pi_G \times \pi_F) & & \pi_G \hat{\otimes} F \\
\downarrow & & \downarrow \\
C & & C
\end{array}
\]

in the category \textbf{Cat} of categories and functors.
A key observation

The category $\text{Cat}/C$ inherits a tensor product

$$\boxtimes : \text{Cat}/C \times \text{Cat}/C \rightarrow \text{Cat}/C$$

from the monoidal structure of the category $C$.

The Day tensor product

$$\hat{\boxtimes} : \text{DiscOpFib}/C \times \text{DiscOpFib}/C \rightarrow \text{DiscOpFib}/C$$

is the monoidal structure obtained by transporting $\boxtimes$ along the adjunction

$$\text{Cat}/C \quad \Downarrow \quad \text{DiscOpFib}/C$$

with the Free and Inclusion diagrams.
The convolution product on double categories

Extending the Day construction
The convolution product on double categories

Given two covariant presheaves

\[ F, G : \mathcal{D}_1 \longrightarrow \text{Set} \]

on a double category \( \mathcal{D} \) with horizontal composition

\[ \diamond_h : \mathcal{D}_2 = \mathcal{D}_1 \times_{\mathcal{D}_0} \mathcal{D}_1 \longrightarrow \mathcal{D}_1 \]

the **convolution product** of \( F \) and \( G \) is the covariant presheaf

\[ G \ast F : \mathcal{D}_1 \longrightarrow \text{Set} \]

defined by the coend formula:

\[
G \ast F = t \mapsto \int_{(s, r) \in \mathcal{D}_2} \mathcal{D}_1(s \diamond_h r, t) \times G(s) \times F(r)
\]
The convolution product

Equivalently, the convolution product

\[ G \ast F : \mathcal{D}_1 \longrightarrow \text{Set} \]

may be defined as the \textbf{left Kan extension} of the functor

\[
\begin{array}{ccc}
\mathcal{D}_1 \times \mathcal{D}_0 \mathcal{D}_1 & \xrightarrow{\text{proj}} & \mathcal{D}_1 \times \mathcal{D}_1 \\
\downarrow \circ h & & \downarrow \psi \ast \varphi \\
\mathcal{D}_1 & \xrightarrow{G \times F} & \text{Set} \times \text{Set}
\end{array}
\]

along the tensor product functor:
What does the coend formula mean?

An element of the coend

\[ G \star F(t) = \int_{(s, r) \in D_2} D_1(s \diamond_h r, t) \times G(s) \times F(r) \]

consists of a double cell of the form

\[
\begin{array}{ccc}
B & \leftarrow & C \\
\uparrow g & & \uparrow \gamma \\
B' & & A' \\
\downarrow t & & \downarrow f \\
& & A
\end{array}
\]

\[ s \]
\[ r \]

\[ B \]
\[ C \]
\[ A \]
\[ B' \]
\[ A' \]

\[ t \]

\[ f \]

\[ g \]

\[ \gamma \]

\[ s \]

\[ r \]

\[ y \in G(s) \]
\[ x \in F(r) \]

considered modulo an equivalence relation noted \( \sim \).
What does the coend formula mean?

We find enlightening to draw the triple

\[
(s \diamond_h r \xrightarrow{\gamma} t, \quad x \in F(r), \quad y \in G(s))
\]

in the following way:

This picture is the reason we like to speak of the rabbit calculus.
What does the coend formula mean?

Suppose given a pair of elements

\[ x \in F(r) \quad \quad y \in G(s) \]

a pair of double cells

\[ \alpha : r \Rightarrow r' \quad \quad \beta : s \Rightarrow s' \]

and a double cell

\[ \gamma : s' \diamond_h r' \Rightarrow t \]
What does the coend formula mean?

The five components may be depicted as follows:
What does the coend formula mean?

The equivalence relation \( \sim \) defined by the coend

\[
G \ast F (t) = \int_{(s, r) \in D_2} \mathbb{D}_1(s \diamond_h r, t) \times G(s) \times F(r)
\]

identifies every triple of the form
Main structural theorem

Theorem [Behr, PAM, Zeilberger]

The convolution product

\[ G, F \mapsto G \ast F \]

on a double category \( \mathcal{D} \) defines a functor

\[ \ast : \widehat{\mathcal{D}} \times \widehat{\mathcal{D}} \to \widehat{\mathcal{D}} \]

which equips the category of covariant presheaves

\[ \widehat{\mathcal{D}} := [\mathcal{D}_1, \text{Set}] \]

with the structure of an **oplax monoidal closed category**.
What oplax monoidal means...

The category of covariant presheaves

\[ \hat{\mathcal{D}} := [\mathcal{D}_1, \text{Set}] \]

comes equipped with a family of convolution products

\[ *_n : \hat{\mathcal{D}} \times \cdots \times \hat{\mathcal{D}} \to \hat{\mathcal{D}} \]

where we use the notation

\[ (F_n \ast \cdots \ast F_1) := *_n (F_n, \ldots, F_1) \]

for the \( n \)-ary product of \( n \) covariant presheaves

\[ F_n, \ldots, F_1 : \mathcal{D}_1 \to \text{Set}. \]
The ternary convolution product

Typically, the ternary convolution product

$$H \ast G \ast F : \quad C \longrightarrow \text{Set}$$

of three covariant presheaves $H, G, F$ is defined by the coend formula

$$H \ast G \ast F = u \mapsto \int_{(t, s, r) \in D_3} D_1(t \odot_h s \odot_h r, u) \times H(t) \times G(s) \times F(r)$$

where $D_3$ is the category of horizontal paths of length 3.
The ternary convolution product

The elements of the ternary convolution product are quadruples

\[
(t \ast_h s \ast_h r \xrightarrow{\delta} u, \quad x \in F(r), \quad y \in G(s), \quad z \in G(t))
\]

which may be depicted in the following way:
The ternary convolution product

The elements of the convolution product

\[(t \diamond_h s \diamond_h r \overset{\delta}{\longrightarrow} u, \quad x \in F(r), \quad y \in G(s), \quad z \in G(t) \]

are identified modulo the equivalence relation:
What oplax monoidal means...

The convolution products are related by associativity maps such as

\[ H \ast (G \ast F) \xleftrightarrow{\text{assoc}} (H \ast G \ast F) \xrightarrow{\text{assoc}} (H \ast G) \ast F \]

which are not reversible in general, for the following reason:
What oplax monoidal means...

In a general double category $\mathcal{D}$, not every composite shape of the form defining an element of the presheaf $H \ast (G \ast F)$ at instance $u : A \to A'$
What oplax monoidal means...

is equivalent modulo $\sim$ in $\mathcal{D}$ to a ternary shape of the form

\[
\begin{array}{ccc}
H & G & F \\
\downarrow z & \downarrow y & \downarrow x \\
\uparrow t & \uparrow s & \uparrow r \\
\mu & \mu & \mu \\
\downarrow k & \downarrow j & \downarrow u \\
\end{array}
\]

defining an element of $H \ast G \ast F$ at the same instance $u : A \to A'$. 
Sketch of the proof

**Key observation.** The projection functor

\[
\pi_{G \ast F} : \text{Elts}(G \ast F) \longrightarrow \mathbb{D}_1
\]

associated to the binary convolution product

\[
G \ast F : \mathbb{D}_1 \longrightarrow \text{Set}
\]

is the **free discrete opfibration** associated to the functor

\[
\text{Elts}(G) \otimes \text{Elts}(F) = \text{Elts}(G) \times_{\mathbb{D}_0} \text{Elts}(F) \quad \overset{\pi_G \otimes \pi_F}{\longrightarrow} \quad \mathbb{D}_2 \quad \overset{h_2}{\longrightarrow} \quad \mathbb{D}_1
\]

obtained by composing the two projection functors above \(\mathbb{D}_1\)

\[
\text{Elts}(G) \quad \overset{\pi_G}{\longrightarrow} \quad \mathbb{D}_1 \quad \quad \quad \text{Elts}(F) \quad \overset{\pi_F}{\longrightarrow} \quad \mathbb{D}_1
\]
Sketch of the proof

\[
\begin{array}{ccc}
\text{Elts } (G) & \xrightarrow{\pi_G} & \mathbb{D}_1 \\
& \xrightarrow{T} & \mathbb{D}_0 \\
\text{Elts } (F) & \xrightarrow{\pi_F} & \mathbb{D}_1 \\
& \xrightarrow{T} & \mathbb{D}_0 \\
& \xrightarrow{S} & \mathbb{D}_0 \\
\end{array}
\]
Sketch of the proof

\[ \text{Elts}(G) \odot \text{Elts}(F) \]

Diagram:

- \( \pi_G \)
- \( T \)
- \( S \)
- \( \pi_F \)

Nodes:

- \( D_0 \)
- \( D_1 \)
Sketch of the proof

\[ \text{Elts}(G) \otimes \text{Elts}(F) \]

\[ \pi_G \otimes \pi_F \]

\[ D_0 \xleftarrow{T} D_1 \xrightarrow{T} D_0 \]
\[ D_0 \xleftarrow{S} D_1 \xrightarrow{S} D_0 \]
\[ D_0 \xleftarrow{D_1} D_1 \xrightarrow{D_1} D_0 \]
Sketch of the proof

\[
\begin{align*}
\text{Elts}(G) \oplus \text{Elts}(F) & \quad \pi_G \circ \pi_F \\
\text{Elts}(G) & \quad \pi_G \quad \text{Elts}(F) \\
\pi_G & \quad T \quad \pi_F \\
\mathbb{D}_2 & \quad h_2 \quad \mathbb{D}_1 \\
\mathbb{D}_1 & \quad T \\
\mathbb{D}_0 & \quad S \\
\mathbb{D}_1 & \quad S \quad \mathbb{D}_0
\end{align*}
\]
Sketch of the proof

Similarly, the projection functor

$$
\pi_{H \star G \star F} : \text{Elts} (H \star G \star F) \rightarrow \mathbb{D}_1
$$

associated to the ternary convolution product

$$
H \star G \star F : \mathbb{D}_1 \rightarrow \text{Set}
$$
is the **free discrete opfibration** associated to the functor

$$
\text{Elts} (H) \otimes \text{Elts} (G) \otimes \text{Elts} (F) \xrightarrow{\pi_H \otimes \pi_G \otimes \pi_F} \mathbb{D}_3 \xrightarrow{h_3} \mathbb{D}_1
$$

obtained by composing the three projection functors above $\mathbb{D}_1$

$$
\text{Elts} (H) \xrightarrow{\pi_H} \mathbb{D}_1 \quad \text{Elts} (G) \xrightarrow{\pi_G} \mathbb{D}_1 \quad \text{Elts} (F) \xrightarrow{\pi_F} \mathbb{D}_1
$$
Sketch of the proof

Elts (H) \[\xrightarrow{\pi_H} D_1\]

\[\xrightarrow{T} D_0\]

Elts (G) \[\xrightarrow{\pi_G} D_1\]

\[\xrightarrow{T} D_0\]

Elts (F) \[\xrightarrow{\pi_F} D_1\]

\[\xrightarrow{S} D_0\]
Sketch of the proof

\[ \text{Elts} (H) \otimes \text{Elts} (G) \otimes \text{Elts} (F) \]

Diagram:

- \( \pi_H \) from \( \text{Elts} (H) \) to \( D_1 \)
- \( \pi_G \) from \( \text{Elts} (G) \) to \( D_1 \)
- \( \pi_F \) from \( \text{Elts} (F) \) to \( D_1 \)
- \( T \) from \( D_1 \) to \( D_0 \)
- \( S \) from \( D_1 \) to \( D_0 \)
Sketch of the proof

\[\text{Elts} (H) \oplus \text{Elts} (G) \oplus \text{Elts} (F)\]
Sketch of the proof

\[
\text{Elts}(H) \otimes \text{Elts}(G) \otimes \text{Elts}(F)
\]

\[
\xymatrix{ & \mathbb{D}_3 \\
\mathbb{D}_0 \ar[ru]^{T} & \mathbb{D}_1 \ar[u]^{\pi_H} \ar[lu]_{T} & \mathbb{D}_1 \ar[u]_{\pi_G} \ar[ru]_{h_3} & \mathbb{D}_0 \\
& \mathbb{D}_1 \ar[u]_{\pi_F} \ar[lu]_{S} }
\]
Main argument of the proof

The category $\text{Cat/} \mathcal{D}_1$ inherits a monoidal structure

$$\otimes : \text{Cat/} \mathcal{D}_1 \times \text{Cat/} \mathcal{D}_1 \longrightarrow \text{Cat/} \mathcal{D}_1$$

computed by pullback using the double categorical structure of $\mathcal{D}$.

The convolution product

$$\ast : \text{DiscOpFib/} \mathcal{D}_1 \times \text{DiscOpFib/} \mathcal{D}_1 \longrightarrow \text{DiscOpFib/} \mathcal{D}_1$$

is the oplax monoidal structure obtained by transporting on $\hat{\mathcal{D}} = [\mathcal{D}_1, \text{Set}]$ the strong monoidal structure $\otimes$ on $\text{Cat/} \mathcal{D}_1$ along the adjunction

$$\begin{array}{ccc}
\text{Cat/} \mathcal{D}_1 & \downarrow & \text{DiscOpFib/} \mathcal{D}_1 \\
\text{Free} & & \text{Inclusion}
\end{array}$$
Cylindrical decomposition property

A sufficient condition to ensure strong associativity
Towards strong associativity

We want to find a **sufficient condition** on a double category

\[(\mathcal{D}, h_n : \mathcal{D}_n \to \mathcal{D}_1)\]

ensuring that the **associativity maps** of the convolution product

\[H \ast (G \ast F) \xleftarrow{\text{assoc}} (H \ast G \ast F) \xrightarrow{\text{assoc}} (H \ast G) \ast F\]

are **reversible**.
Towards strong associativity

In particular, this requires to show that every composite shape defining an element of the presheaf $H \ast (G \ast F)$ at instance $u : A \rightarrow A'$
Towards strong associativity

is equivalent modulo $\sim$ in $\mathbb{D}$ to a ternary shape of the form

$$H \rightarrow A \rightarrow A' \leftarrow F$$

defining an element of $H \ast G \ast F$ at the same instance $u: A \rightarrow A'$. 
Towards strong associativity

Suppose that every double cell of the form

\[
\begin{array}{c}
\gamma \\
\end{array}
\]

factors in the following way:

\[
\begin{array}{c}
\gamma \\
\end{array} = \begin{array}{c}
\beta \\
\alpha \\
\end{array}
\]
Towards strong associativity

In that case, one can rewrite the original composite shape
Towards strong associativity

We then into the shape where the cell $\gamma$ has been factored:
Towards strong associativity

then into the equivalent shape using the equivalence relation $\sim$
Towards strong associativity

then into the equal shape by vertical composition:
Towards strong associativity

and finally in the ternary shape we were looking for:
The cylinder categories

Every double category $\mathcal{D}$ comes equipped with a family of categories $\text{Cyl}_{\mathcal{D}}[n]$ called \textbf{cylinder categories} and defined in the following way:

- the objects of $\text{Cyl}_{\mathcal{D}}[n]$ are the tuples
  \[\sigma = (s_n, \ldots, s_1, s, \sigma : s_n \diamond h \cdots \diamond h s_1 \Rightarrow s)\]

defining a \textbf{globular cell} of the form

\[
\begin{align*}
    A_n & \xleftarrow{s_n} A_{n-1} \cdots A_4 \xleftarrow{s_3} A_3 \xleftarrow{s_2} A_2 \xleftarrow{s_1} A_1 \\
    \downarrow id & \quad \downarrow id \\
    A_n & \quad \quad A_1 \\
    \downarrow & \quad \downarrow \\
    \quad & \quad \\
    & \quad \\
\end{align*}
\]
The cylinder categories

given globular cells

\[
\sigma = (s_n, \ldots, s_1, s, \sigma : s_n \diamond_h \cdots \diamond_h s_1 \Rightarrow s)
\]

\[
\tau = (t_n, \ldots, t_1, t, \tau : t_n \diamond_h \cdots \diamond_h t_1 \Rightarrow t)
\]

the morphisms of \( \textbf{Cyl}_D[n] \) of the form

\[
(\varphi_n, \ldots, \varphi_1, \varphi) : \sigma \longrightarrow \tau
\]

are tuples consisting of a map in \( D_n \)

\[
(\varphi_n, \ldots, \varphi_1) : (s_n, \ldots, s_1) \Rightarrow (t_n, \ldots, t_1)
\]

and of a double cell

\[
\varphi : s \Rightarrow t
\]
The cylinder categories

such that the double cell \( \varphi \circ \sigma \) depicted below

\[
\begin{array}{cccccccc}
A_n & \leftarrow & A_{n-1} & \cdots & A_3 & \leftarrow & A_2 & \leftarrow & A_1 & \leftarrow & A_0 \\
\downarrow^{id} & & \downarrow^{\sigma} & & \downarrow^{\varphi} & & \downarrow^{id} & & \\
A_n & & & & & & & & & \\
\downarrow^{a_n} & & \downarrow^{\varphi} & & \downarrow^{a_0} & & \\
B_n & \leftarrow & & & & & & & & & \\
\downarrow & & \downarrow & & \downarrow & & & & & \\
B_0 & & & & & & & & &
\end{array}
\]
The cylinder categories is equal to the double cell $\tau \circ (\varphi_n \circ_h \cdots \circ_h \varphi_1)$ depicted below:

\[
\begin{array}{cccccccc}
\cdots & A_3 & A_2 & A_1 & A_0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\varphi_3 & a_2 & \varphi_2 & a_1 & \varphi_1 & a_0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
B_3 & B_2 & B_1 & B_0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\tau & \tau & \tau & \tau \\
\downarrow & \downarrow & \downarrow & \downarrow \\
B_0 & B_0 & B_0 & B_0 \\
\end{array}
\]
The cylindrical decomposition property

Key observation: each composition functor

\[ h_n : \mathcal{D}_n \rightarrow \mathcal{D}_1 \]

of the double category \( \mathcal{D} \) factors as

\[ \mathcal{D}_n \rightarrow \text{Cyl}_{\mathcal{D}}[n] \overset{\pi_n}{\rightarrow} \mathcal{D}_1 \]

Definition. A double category \( \mathcal{D} \) satisfies the \( n \)-cylindrical decomposition property (\( n \)-CDP)

when the functor

\[ \text{Cyl}_{\mathcal{D}}[n] \overset{\pi_n}{\rightarrow} \mathcal{D}_1 \]

is an opfibration.
Main theorem

Theorem. [Behr,PAM,Zeilberger]

Suppose that a double category $\mathcal{D}$ satisfies

the $n$-cylindrical decomposition property ($n$-CDP)

for all $n \in \mathbb{N}$.

In that case, the convolution product defines a functor

$$\ast : \hat{\mathcal{D}} \times \hat{\mathcal{D}} \to \hat{\mathcal{D}}$$

which equips the category of covariant presheaves

$$\hat{\mathcal{D}} := [\mathcal{D}_1, \text{Set}]$$

with the structure of a strong monoidal closed category.
Main theorem

In particular, the associativity maps are reversible in that case:

\[ H \ast (G \ast F) \xleftarrow{\text{assoc}} (H \ast G \ast F) \xrightarrow{\text{assoc}} (H \ast G) \ast F \]

Reversibility comes from the cylindrical decomposition property of $D$. 
Illustrations

The theorem applies to the following situations:

- every bicategory $\mathcal{D} = \mathcal{W}$ satisfies $n$-CDP,

- every framed bicategory $\mathcal{D} = \mathcal{W}$ satisfies $n$-CDP for $n \geq 1$,

- the double category $\mathcal{D} = \mathcal{DPO}$ satisfies $n$-CDP for $n \geq 1$.

More generally, the theorem enables us to use the convolution product for a number of categorical graph rewriting frameworks.
Categorifying rule algebras

Composing representable presheaves by convolution
Categorification of rule algebras

One main ingredient of rule algebras is the following equation

\[\delta(r) \star \delta(s) = \sum_{\mu \in M_r(s)} \delta(r_{\mu}s)\]

where

- \(M_r(s)\) is the set of admissible matches of rule \(r\) into rule \(s\)
- \(r_{\mu}s\) denotes one possible way to get a composite rule from \(r\) and \(s\).

Similarly, we want to find sufficient conditions on \(D\) such that

\[\hat{\Delta}_r \star \hat{\Delta}_s = \sum_{\mu \in M_r(s)} \hat{\Delta}_{r_{\mu}s}\]

where the sum is now set-theoretic union.
Multi-sums

Suppose that $A$ and $B$ are objects in a category $C$.

**Definition.** A **multi-sum** of $A$ and $B$ is a family of cospans

\[
( A \xrightarrow{a_i} U_i \xleftarrow{b_i} B )_{i \in I}
\]

such that for any cospan

\[
A \xrightarrow{f} X \xleftarrow{g} B
\]

there exists a unique $i \in I$ and a unique morphism

\[
[f, g] : U_i \xrightarrow{f} X
\]

such that

\[
f = [f, g] \circ a_i \quad \text{and} \quad g = [f, g] \circ b_i.
\]
Categorification of rule algebras

Assume $\mathcal{D}$ is a small double category satisfying

- the vertical category $\mathcal{D}_0$ has multi-sums,
- the source and target functors $S, T : \mathcal{D}_1 \to \mathcal{D}_0$ are opfibrations.

In that case, the convolution product of two representable presheaves is isomorphic to the sum of representables

$$\hat{\Delta}_{r_2} \ast \hat{\Delta}_{r_1} \cong \sum_{i \in I} \hat{\Delta}_{r_2 \langle c_i \rangle \circ \langle b_i \rangle r_1}$$

where the multi-sum of $B$ and $C$ is given by a family of cospans

$$\left( B \xrightarrow{b_i} U_i \xleftarrow{c_i} C \right)_{i \in I}$$

and where $r_2 \langle c_i \rangle$ denotes the $S$-pushforward of $r_2$ along $c_i$ and $\langle b_i \rangle r_1$ denotes the $T$-pushforward of $r_1$ along $b_i$. 
Thank you!