The rabbit calculus:
convolution products on double categories
and categorification of rule algebra

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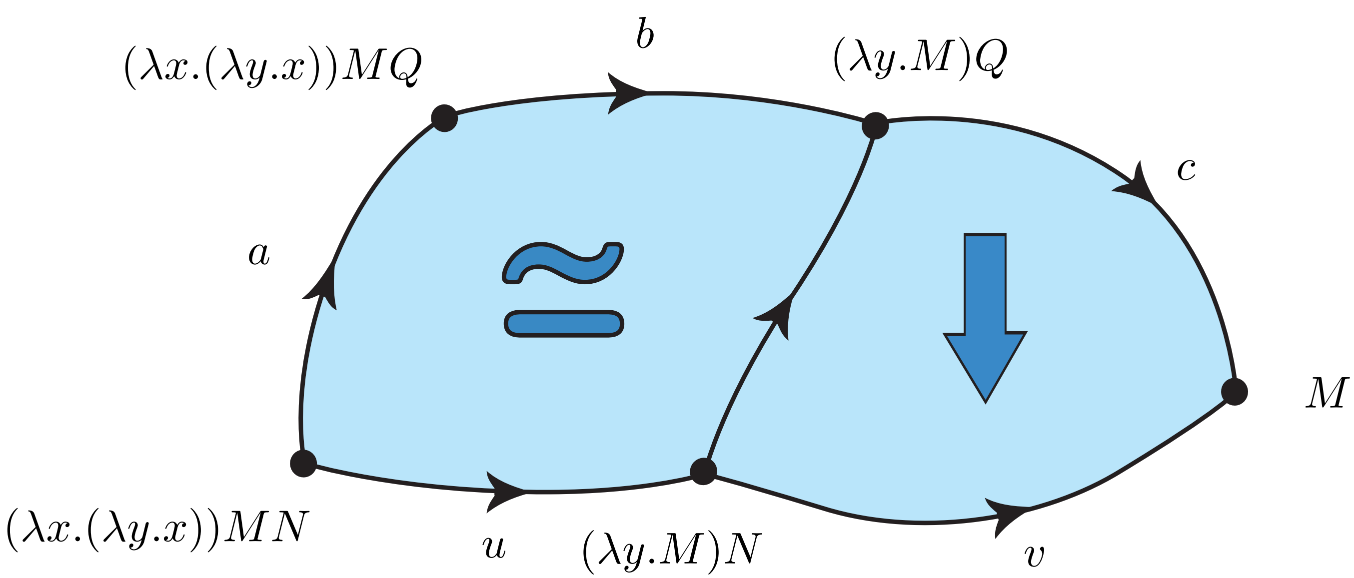
The quest for causality in rewriting theory

An important insight coming from Huet and Lévy:

In order to track the **causality structure** relating different β-redexes, one needs to consider rewriting paths modulo **permutations** of the form



The quest for causality in rewriting theory

In the λ-calculus and term rewriting systems

A tradition based on **optimality** and **residual theory**

- ▷ the notion of **Lévy families** in the λ-calculus (Lévy 1980)
- ▷ their generalisation to any CRS (Asperti, Laneve 1995)
- ▷ a residual theory based on the notion of trek (PAM, 2002)

More recently, in categorical graph rewriting

- ▷ the notion of **tracelet** emerging in the work by Nicolas Behr.

Our ambition in this work is to initiate a convergence between these lines by revisiting/categorifying the work on tracelets using **double categories.**

Double categories

Definition. A (weak) **double category** $\mathbb{D}$ consists of

- ▷ a category $\mathbb{D}_0$ of objects,
- ▷ a category $\mathbb{D}_1$ of horizontal maps,
- ▷ a pair of **source** and **target** functors

$$\mathbb{D}_0 \xleftarrow{\quad T \quad} \mathbb{D}_1 \xrightarrow{\quad S \quad} \mathbb{D}_0$$

- ▷ a **horizontal composition** functor

$$\diamond_h \quad : \quad \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \longrightarrow \mathbb{D}_1$$

- ▷ a **horizontal identity** functor

$$idh \quad : \quad \mathbb{D}_0 \longrightarrow \mathbb{D}_1$$

satisfying a number of **associativity** and **neutrality** properties.

The category $\mathbb{D}_0$ of vertical maps

A morphism in the category $\mathbb{D}_0$ is represented as a **vertical map**

$$\begin{tikzcd} A \arrow[d, "a"'] \\ A' \end{tikzcd}$$

which may be **composed vertically** with other vertical maps.

The category $\mathbb{D}_1$ of horizontal maps

An object in the category $\mathbb{D}_1$ is represented as a **horizontal map**

$$B \xleftarrow{r} A$$

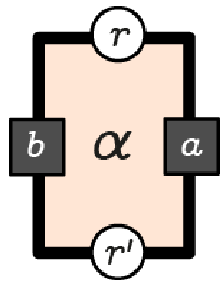
A morphism in the category $\mathbb{D}_1$ is represented as a **double cell**

$$\begin{tikzcd} B \arrow[d, "b"'] & A \arrow[l, "r"'] \arrow[d, "a"] \arrow[dl, phantom, "\Downarrow \alpha"] \\ B' & A' \arrow[l, "r'"] \end{tikzcd}$$

which may be **composed vertically** with other double cells.

The category $\mathbb{D}_1$ of horizontal maps

We often find convenient to use the pictorial notation



for the double cell usually noted

$$\begin{tikzcd} B \arrow[d, "b"'] & A \arrow[l, "r"'] \arrow[d, "a"] \arrow[dl, phantom, "\Downarrow\alpha"] \\ B' & A' \arrow[l, "r'"] \end{tikzcd}$$

The category $\mathbb{D}_2$ of paths of length 2

Every double category $\mathbb{D}$ comes with

a category $\mathbb{D}_2 = \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1$ of horizontal paths of length 2

defined as the limit of the diagram of functors

$$\begin{tikzcd}
 & & \mathbb{D}_2 \arrow[dl] \arrow[dr] & & \\
 & \mathbb{D}_1 \arrow[dl, "T"'] \arrow[dr, "S"] & & \mathbb{D}_1 \arrow[dl, "T"'] \arrow[dr, "S"] & \\
\mathbb{D}_0 & & \mathbb{D}_0 & & \mathbb{D}_0
\end{tikzcd}$$

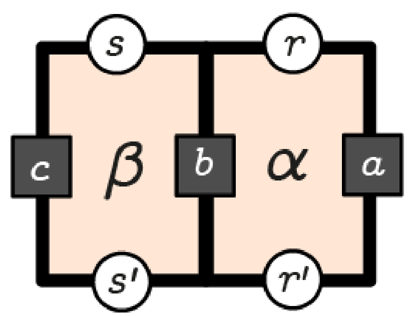
in the category $\mathbf{Cat}$ of categories and functors.

The category $\mathbb{D}_2$ of paths of length 2

A typical morphism of $\mathbb{D}_2$ has the shape

$$
\begin{tikzcd}
C \arrow[d, "c"'] & & B \arrow[ll, "s"'] \arrow[d, "b"'] \arrow[ll, phantom, shift right=8ex, "\Downarrow\beta"] & & A \arrow[ll, "r"'] \arrow[d, "a"] \arrow[ll, phantom, shift right=8ex, "\Downarrow\alpha"] \\
C' & & B' \arrow[ll, "s'"] & & A' \arrow[ll, "r'"]
\end{tikzcd}
$$

which we also like to depict as



The category $\mathbb{D}_3$ of paths of length 3

Every double category $\mathbb{D}$ comes with

a category $\mathbb{D}_3$ of horizontal paths of length 3

defined as the limit of the diagram of functors

$$\begin{tikzcd}[column sep=small]
& & & \mathbb{D}_3 \arrow[dll] \arrow[d] \arrow[drr] & & & \\
& \mathbb{D}_1 \arrow[dl, "T"'] \arrow[dr, "S"] & & \mathbb{D}_1 \arrow[dl, "T"'] \arrow[dr, "S"] & & \mathbb{D}_1 \arrow[dl, "T"'] \arrow[dr, "S"] & \\
\mathbb{D}_0 & & \mathbb{D}_0 & & \mathbb{D}_0 & & \mathbb{D}_0
\end{tikzcd}$$

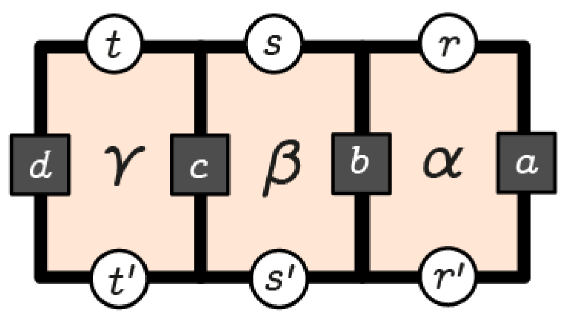
in the category $\mathbf{Cat}$ of categories and functors.

The category $\mathbb{D}_3$ of paths of length 3

A typical morphism of $\mathbb{D}_3$ has the shape

$$\begin{tikzcd}
D \arrow[d, "d"'] & C \arrow[l, "t"'] \arrow[d, "c"'] \arrow[dl, phantom, "\Downarrow\gamma"] & B \arrow[l, "s"'] \arrow[d, "b"] \arrow[dl, phantom, "\Downarrow\beta"] & A \arrow[l, "r"'] \arrow[d, "a"] \arrow[dl, phantom, "\Downarrow\alpha"] \\
D' & C' \arrow[l, "t'"] & B' \arrow[l, "s'"] & A' \arrow[l, "r'"]
\end{tikzcd}$$

which we also like to depict as



The category $\mathbb{D}_4$ of paths of length 4

Every double category $\mathbb{D}$ comes with

a category $\mathbb{D}_4$ of horizontal paths of length 4

defined as the limit of the diagram of functors

$$\begin{tikzcd}
& & & & \mathbb{D}_4 \arrow[dlll] \arrow[dl] \arrow[dr] \arrow[drrr] & & & & \\
& \mathbb{D}_1 \arrow[dl, "T"'] \arrow[dr, "S"] & & \mathbb{D}_1 \arrow[dl, "T"'] \arrow[dr, "S"] & & \mathbb{D}_1 \arrow[dl, "T"'] \arrow[dr, "S"] & & \mathbb{D}_1 \arrow[dl, "T"'] \arrow[dr, "S"] & \\
\mathbb{D}_0 & & \mathbb{D}_0 & & \mathbb{D}_0 & & \mathbb{D}_0 & & \mathbb{D}_0
\end{tikzcd}$$

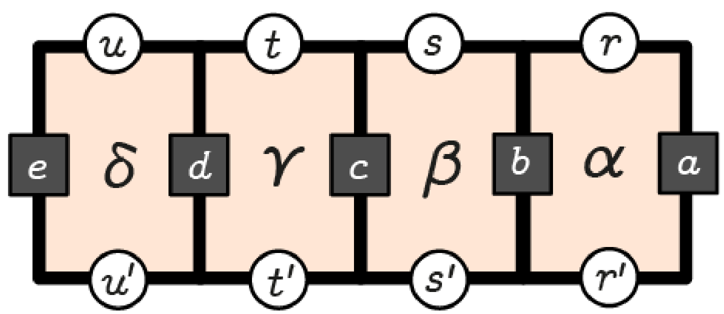
in the category $\mathbf{Cat}$ of categories and functors.

The category $\mathbb{D}_4$ of paths of length 4

A typical morphism of $\mathbb{D}_4$ has the shape

$$\begin{tikzcd}
E \arrow[d, "e"'] & D \arrow[l, "u"'] \arrow[d, "d"'] \arrow[dl, phantom, "\Downarrow\delta"] & C \arrow[l, "t"'] \arrow[d, "c"'] \arrow[dl, phantom, "\Downarrow\gamma"] & B \arrow[l, "s"'] \arrow[d, "b"] \arrow[dl, phantom, "\Downarrow\beta"] & A \arrow[l, "r"'] \arrow[d, "a"] \arrow[dl, phantom, "\Downarrow\alpha"] \\
E' & D' \arrow[l, "u'"] & C' \arrow[l, "t'"] & B' \arrow[l, "s'"] & A' \arrow[l, "r'"]
\end{tikzcd}$$

which we also like to depict as



Unbiased presentation of a double category

Every double category $\mathbb{D}$ comes equipped with a family of functors

$$h_n \quad : \quad \mathbb{D}_n \longrightarrow \mathbb{D}_1$$

called the **horizontal composition** functors, and satisfying a number of **associativity** and **neutrality** properties.

This leads to an alternative (unbiased) definition of (weak) double category.

Note that the functors h_2 and h_0 coincide with the functors $\diamond_h$ and idh

$$h_2 = \diamond_h \quad : \quad \mathbb{D}_2 \longrightarrow \mathbb{D}_1$$

$$h_0 = idh \quad : \quad \mathbb{D}_0 \longrightarrow \mathbb{D}_1$$

The double category $\mathbf{DPO}$ of double pushouts

The double category $\mathbb{D} = \mathbf{DPO}$ on an adhesive category $\mathbf{C}$

- ▷ whose objects are objects A, B, C of the cohesive category $\mathbf{C}$,
- ▷ whose horizontal maps $M = (S, s, t)$ are spans in $\mathbf{C}$,
- ▷ whose vertical maps $\lambda_A : A \to A'$ are monos in $\mathbf{C}$,
- ▷ whose double cells $\theta : M \Rightarrow M'$ are monos $\lambda_\theta : S \to S'$ making the pushout diagram commute:

$$
\begin{tikzcd}
B \arrow[d, tail, "\lambda_B"'] & A \arrow[l, "M"', "|"] \arrow[d, tail, "\lambda_A"] \arrow[dl, phantom, "\Downarrow\theta"] \\
B' & A' \arrow[l, "M'", "|"']
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[d, tail, "\lambda_B"'] \arrow[dr, phantom, "PO"] & S \arrow[l, "t"'] \arrow[r, "s"] \arrow[d, tail, "\lambda_\theta"] \arrow[dr, phantom, "PO"] & A \arrow[d, tail, "\lambda_A"] \\
B' & S' \arrow[l, "t'"] \arrow[r, "s'"'] & A'
\end{tikzcd}
$$

Rewriting rules as covariant presheaves

A **rewriting rule** provided by a **horizontal map**

$$r \quad : \quad B \longleftarrow A$$

is described in our framework as the **representable presheaf**

$$\hat{\Delta}_r \quad : \quad \mathbb{D}_1 \longrightarrow \mathbf{Set}$$

which associates to every horizontal map

$$u \quad : \quad B' \longleftarrow A'$$

the set

$$\mathbb{D}_1(r, u)$$

of all possible **implementations** of the transformation u by the rule r.

Category of elements of a presheaf

The Grothendieck construction

Elements of a covariant presheaf

Recall that an **element**

$$(a, x) \quad \in \quad \mathbf{Elts}\,(F)$$

of a covariant presheaf

$$F \quad : \quad \mathbf{C} \longrightarrow \mathbf{Set}$$

is defined as a pair

$$\Big(\quad a \in \mathbf{C} \quad , \quad x \in F(a) \quad \Big)$$

consisting of

- ▷ an object a of the underlying category $\mathbf{C}$,
- ▷ an element x of the set $F(a)$.

Elements of a covariant presheaf

We find enlightening to draw such a pair

$$\Big(\quad a \in \mathbf{C} \quad , \quad x \in F(a) \quad \Big) \quad \in \quad \mathbf{Elts}\,(F)$$

in the following way

$$\begin{tikzcd} F \\ a \arrow[u, dashed, dash, "x"'] \end{tikzcd}$$

with the intuition that the element

$$x \in F(a)$$

provides a **witness** of the covariant presheaf F at instance $a \in \mathbf{C}$.

Covariant action of a presheaf

By definition of a covariant presheaf

$$F \quad : \quad \mathbf{C} \longrightarrow \mathbf{Set}$$

every element

$$\Big(\quad a \in \mathbf{C} \quad , \quad x \in F(a) \quad \Big) \quad \in \quad \mathbf{Elts}\,(F)$$

and morphism of the category $\mathbf{C}$

$$\gamma \quad : \quad a \longrightarrow a'$$

induces an element

$$\Big(\quad a' \in \mathbf{C} \quad , \quad \gamma \cdot x = F(\gamma)(x) \in F(a') \quad \Big) \quad \in \quad \mathbf{Elts}\,(F)$$

Covariant action of a presheaf

This means that every diagram

$$\begin{tikzcd} & F & \\ a \arrow[ur, dash, dashed, "x"] \arrow[rr, "\gamma"] & & a' \end{tikzcd}$$

can be completed into the diagram

$$\begin{tikzcd} & F & \\ a \arrow[ur, dash, dashed, "x"] \arrow[rr, "\gamma"] & & a' \arrow[ul, dash, dashed, "\gamma\cdot x = F(\gamma)(x)"'] \end{tikzcd}$$

The category of elements

The category $\mathbf{Elts}\,(F)$ of elements of a covariant presheaf

$$F \quad : \quad \mathbf{C} \longrightarrow \mathbf{Set}$$

is defined in the following way:

- ▷ its objects are the elements (a, x) of the covariant presheaf F
- ▷ its morphisms

$$(f, x) \quad : \quad (a, x) \longrightarrow (a', x')$$

are the pairs consisting of a morphism

$$f \quad : \quad a \longrightarrow a'$$

of the category $\mathbf{C}$ and an element $x \in F(a)$ such that

$$f \cdot x \quad = \quad F(f)(x) \quad = \quad x'$$

The category of elements

The category of elements

$$\mathbf{Elts}\,(F)$$

associated to a covariant presheaf

$$F \quad : \quad \mathbf{C} \longrightarrow \mathbf{Set}$$

comes equipped with a **projection functor**

$$\pi_F \quad : \quad \mathbf{Elts}\,(F) \longrightarrow \mathbf{C}$$

which transports every element

$$(a, x) \in \mathbf{Elts}\,(F)$$

to the object $a \in \mathbf{C}$ of the underlying category $\mathbf{C}$.

Fact. The functor π_F defines a **discrete opfibration**.

Grothendieck opfibrations

Definition. A functor

$$p \quad : \quad \mathbf{E} \longrightarrow \mathbf{C}$$

is an **opfibration** when there exists an opcartesian morphism

$$\begin{tikzcd} R \arrow[r, "f"] & S \\ A \arrow[r, "u"'] & B \end{tikzcd} \qquad \text{with } p \text{ mapping the top row } (\mapsto) \text{ to the bottom row}$$

for every object $R \in p^{-1}(A)$ and every morphism $u : A \to B$.

Opcartesian morphisms

A morphism $f : R \to S$ in $\mathbf{E}$ is opcartesian above $u : A \to B$ in $\mathbf{C}$ when the following property holds:

for every map $g : R \to T$

for every map $v : B \to C$
such that $p\,(g) = v \circ u$

there exists
a unique map $h : S \to T$
such that $h \circ f = g$
and $p\,(h) = v$.

$$\begin{tikzcd}
R \arrow[r, "f"'] \arrow[rr, bend left, "g"] & S \arrow[r, dotted, "h"'] & T
\end{tikzcd}$$

$$\Big\downarrow p$$

$$\begin{tikzcd}
A \arrow[r, "u"'] \arrow[rr, bend left] & B \arrow[r, "v"'] & C
\end{tikzcd}$$

The Grothendieck correspondence

The projection functor

$$\pi_F \quad : \quad \mathbf{Elts}\,(F) \longrightarrow \mathbf{C}$$

is a discrete opfibration. Indeed, every diagram

$$\begin{tikzcd} x \arrow[d, mapsto, "\pi_F"'] & \\ a \arrow[r, "f"] & a' \end{tikzcd}$$

can be completed with the opcartesian morphism $(\,f\,,\,x\,)$ as follows:

$$\begin{tikzcd} x \arrow[r, "(\,f\,,x\,)\,\in\,\mathbf{Elts}\,(F)"] \arrow[d, mapsto, "\pi_F"'] & f \cdot x \arrow[d, mapsto, "\pi_F"] \\ a \arrow[r, "f\,\in\,\mathbf{C}"] & a' \end{tikzcd}$$

The Grothendieck correspondence

Moreover, every natural transformation

$$\begin{tikzcd} \mathbf{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathbf{Set} \arrow[Rightarrow, from=U, to=D, "\theta"] \end{tikzcd}$$

induces a commutative diagram of discrete opfibrations:

$$\begin{tikzcd} \mathbf{Elts}\,(F) \arrow[rr, "\mathbf{Elts}\,(\theta)"] \arrow[dr, "\pi_F"'] & & \mathbf{Elts}\,(G) \arrow[dl, "\pi_G"] \\ & \mathbf{C} & \end{tikzcd}$$

The Grothendieck correspondence

Fact. This induces a categorical equivalence between

- ▹ The category $[\mathbf{C}, \mathbf{Ens}]$ of **covariant presheaves**

$$F, G \quad : \quad \mathbf{C} \longrightarrow \mathbf{Set}$$

  and natural transformations between them.

- ▹ The slice category $\mathbf{DiscOpFib}/\mathbf{C}$ of **discrete opfibrations** above $\mathbf{C}$.

Moreover, there is an adjunction

$$\begin{tikzcd} \mathbf{Cat}/\mathbf{C} \arrow[rr, bend left, "Free"] \arrow[rr, phantom, "\perp"] & & \mathbf{DiscOpFib}/\mathbf{C} \arrow[ll, bend left, "Inclusion"] \end{tikzcd}$$

The Day convolution product

A construction on monoidal categories

The Day convolution product

Given two covariant presheaves

$$F, G \quad : \quad \mathbf{C} \longrightarrow \mathbf{Set}$$

on a monoidal category $\mathbf{C}$ with tensor product

$$\otimes \quad : \quad \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C}$$

the **Day convolution product** of F and G is the covariant presheaf

$$G \,\hat{\otimes}\, F \quad : \quad \mathbf{C} \longrightarrow \mathbf{Set}$$

defined by the coend formula

$$G \,\hat{\otimes}\, F \;=\; c \;\mapsto\; \int^{(b,a)\,\in\,\mathbf{C}\times\mathbf{C}} \mathbf{C}(b \otimes a, c) \times G(b) \times F(a)$$

The Day convolution product

Equivalently, the convolution product

$$G \mathbin{\hat{\otimes}} F \quad : \quad \mathbf{C} \longrightarrow \mathbf{Set}$$

may be defined as the **left Kan extension** of the functor

$$\mathbf{C} \times \mathbf{C} \xrightarrow{\ G \times F\ } \mathbf{Set} \times \mathbf{Set} \xrightarrow{\ \times\ } \mathbf{Set}$$

along the tensor product functor:

$$\begin{tikzcd}
\mathbf{C} \times \mathbf{C} \arrow[r, "G \times F"] \arrow[dr, "\otimes"'] & \mathbf{Set} \times \mathbf{Set} \arrow[r, "\times"] \arrow[d, phantom, "\Downarrow"] & \mathbf{Set} \\
& \mathbf{C} \arrow[ur, "G \hat{\otimes} F"'] &
\end{tikzcd}$$

What does the coend formula mean?

An element of the coend

$$G \hat{\otimes} F\ (c) \quad = \quad \int^{(b,a) \in \mathbf{C} \times \mathbf{C}} \mathbf{C}(b \otimes a, c) \times G(b) \times F(a)$$

consists of a morphism

$$b \otimes a \xrightarrow{\gamma} c$$

together with a pair of elements

$$y \in G(b) \qquad\qquad x \in F(a)$$

considered modulo an equivalence relation $\sim$.

What does the coend formula mean?

As we did before, we find enlightening to draw the two elements

$$y \in G(b) \qquad\qquad x \in F(a)$$

in the following way:

What does the coend formula mean?

Accordingly, we like to draw the triple

$$\left(\quad b\otimes a \xrightarrow{\gamma} c \quad , \quad x \in F(a) \quad , \quad y \in G(b) \quad\right)$$

in the following way:

$$\begin{tikzcd}
G \arrow[dr, dash, dashed, "y"'] & & F \arrow[dl, dash, dashed, "x"] \\
& b\otimes a \arrow[d, "\gamma"] & \\
& c &
\end{tikzcd}$$

What does the coend formula mean?

Suppose given a pair of elements

$$x \in F(a) \qquad\qquad y \in G(b)$$

a pair of morphisms

$$\alpha \; : \; a \longrightarrow a' \qquad\qquad \beta \; : \; b \longrightarrow b'$$

and a morphism

$$\gamma \; : \; a' \otimes b' \longrightarrow c$$

What does the coend formula mean?

The situation may be depicted as follows:

$$
\begin{tikzcd}
G \arrow[dr, dash, dashed, "y"] & & F \arrow[dl, dash, dashed, "x"'] \\
& b \otimes a \arrow[d, "\beta \otimes \alpha"] & \\
& b' \otimes a' \arrow[d, "m"] & \\
& c &
\end{tikzcd}
$$

What does the coend formula mean?

The diagram may be completed as follows:

$$\begin{tikzcd}
G \arrow[dr, dashed, dash, "y"] \arrow[ddr, dashed, dash, bend right, "\beta\cdot y"'] & & F \arrow[dl, dashed, dash, "x"'] \arrow[ddl, dashed, dash, bend left, "\alpha\cdot x"] \\
& b \otimes a \arrow[d, "\beta\otimes\alpha"] & \\
& b' \otimes a' \arrow[d, "\gamma"] & \\
& c &
\end{tikzcd}$$

What does the coend formula mean?

This equivalence relation $\sim$ defined by the coend

$$G \hat{\otimes} F\,(c) \quad = \quad \int^{(b,a)\in \mathbf{C}\times\mathbf{C}} \mathbf{C}(b\otimes a, c) \times G(b) \times F(a)$$

identifies every triple of the form

$$\left(\quad b\otimes a \xrightarrow{\beta\otimes\alpha} b'\otimes a' \xrightarrow{\gamma} c \quad , \quad x \in F(a) \quad , \quad y \in G(b) \quad\right)$$

with the corresponding triple

$$\left(\quad b'\otimes a' \xrightarrow{\gamma} c \quad , \quad \alpha\cdot x \in F(a') \quad , \quad \beta\cdot y \in G(b') \quad\right)$$

What does the coend formula mean?

Diagrammatically, the equivalence relation $\sim$ identifies the two triples:

$$
\begin{tikzcd}
G \arrow[dr, dash, dashed, "y"'] & & F \arrow[dl, dash, dashed, "x"] \\
& b \otimes a \arrow[d, "\beta \otimes \alpha"] & \\
& b' \otimes a' \arrow[d, "\gamma"] & \\
& c &
\end{tikzcd}
\quad \sim \quad
\begin{tikzcd}
G \arrow[dr, dash, dashed, "\beta \cdot y"'] & & F \arrow[dl, dash, dashed, "\alpha \cdot x"] \\
& b' \otimes a' \arrow[d, "\gamma"] & \\
& c &
\end{tikzcd}
$$

The Day convolution product

Theorem [Day 1970] The convolution product

$$G, F \quad \mapsto \quad G \,\hat{\otimes}\, F$$

on a monoidal category $\mathbf{C}$ with tensor product $\otimes$ defines a functor

$$\hat{\otimes} \quad : \quad [\mathbf{C}, \mathbf{Set}] \times [\mathbf{C}, \mathbf{Set}] \longrightarrow [\mathbf{C}, \mathbf{Set}]$$

which equips the category of covariant presheaves

$$[\mathbf{C}, \mathbf{Set}]$$

with the structure of a monoidal closed category.

In particular, the convolution product is associative:

$$H \,\hat{\otimes}\, (G \,\hat{\otimes}\, F) \quad \cong \quad (H \,\hat{\otimes}\, G) \,\hat{\otimes}\, F$$

A key observation

Fact. The projection functor

$$\pi_{G \hat{\otimes} F} \quad : \quad \mathbf{Elts}\,(G \hat{\otimes} F) \longrightarrow \mathbf{C}$$

associated to the Day convolution product

$$G \hat{\otimes} F \quad : \quad \mathbf{C} \longrightarrow \mathbf{Set}$$

is the **free discrete opfibration** associated to the functor

$$\mathbf{Elts}\,(G) \times \mathbf{Elts}\,(F) \xrightarrow{\pi_G \times \pi_F} \mathbf{C} \times \mathbf{C} \xrightarrow{\otimes} \mathbf{C}$$

obtained by tensoring the two projection functors

$$\mathbf{Elts}\,(G) \xrightarrow{\pi_G} \mathbf{C} \qquad\qquad \mathbf{Elts}\,(F) \xrightarrow{\pi_F} \mathbf{C}$$

Construction of the free discrete opfibration

Step 0. We start from the functor

$$\mathbf{Elts}\,(G) \times \mathbf{Elts}\,(F) \xrightarrow{\pi_G \times \pi_F} \mathbf{C} \times \mathbf{C} \xrightarrow{\quad\otimes\quad} \mathbf{C}$$

whose objects in the source category are pairs

$$\Big(\quad x \in F(a) \quad , \quad y \in G(b) \quad \Big)$$

may be depicted in the following way:

$$\begin{tikzcd} G \arrow[dr, dash, dashed, "y"] & & F \arrow[dl, dash, dashed, "x"'] \\ & b \otimes a & \end{tikzcd}$$

Construction of the free discrete opfibration

Step 1. We replace the functor by its **free split opfibration**

$$\mathbf{Elts}\,(G,F) \xrightarrow{\quad \pi_{G,F} \quad} \mathbf{C}$$

where the source category $\mathbf{Elts}\,(G,F)$ has objects defined as triples

$$\Big(\quad b \otimes a \xrightarrow{\gamma} c \quad , \quad x \in F(a) \quad , \quad y \in G(b) \quad \Big)$$

which may be depicted in the following way:

$$\begin{tikzcd}
G \arrow[dr, dashed, no head, "y"'] & & F \arrow[dl, dashed, no head, "x"] \\
& b \otimes a \arrow[d, "\gamma"] & \\
& c &
\end{tikzcd}$$

Construction of the free discrete opfibration

Step 1. We replace the functor by its **free split opfibration**

$$\mathbf{Elts}\,(G,F) \xrightarrow{\quad \pi_{G,F} \quad} \mathbf{C}$$

whose morphisms in each fiber above $c \in \mathbf{C}$ are of the form:

$$\begin{tikzcd} G \arrow[dr, dash, dashed, "y"'] & & F \arrow[dl, dash, dashed, "x"] \\ & b \otimes a \arrow[d, "\beta \otimes \alpha"] & \\ & b' \otimes a' \arrow[d, "\gamma"] & \\ & c & \end{tikzcd} \longrightarrow \begin{tikzcd} G \arrow[dr, dash, dashed, "\beta\cdot y"'] & & F \arrow[dl, dash, dashed, "\alpha\cdot x"] \\ & b' \otimes a' \arrow[d, "\gamma"] & \\ & c & \end{tikzcd}$$

Construction of the free discrete opfibration

Step 2. Replace each fiber category of the opfibration

$$\mathbf{Elts}\,(G,F) \xrightarrow{\quad \pi_{G,F} \quad} \mathbf{C}$$

by its set of **connected components**, using the equivalence relation:

$$
\begin{tikzcd}[column sep=small]
G \arrow[dr, dash, dashed, "y"'] & & F \arrow[dl, dash, dashed, "x"] \\
& b \otimes a \arrow[d, "\beta \otimes \alpha"] & \\
& b' \otimes a' \arrow[d, "\gamma"] & \\
& c &
\end{tikzcd}
\qquad \sim \qquad
\begin{tikzcd}[column sep=small]
G \arrow[dr, dash, dashed, "\beta \cdot y"'] & & F \arrow[dl, dash, dashed, "\alpha \cdot x"] \\
& b' \otimes a' \arrow[d, "\gamma"] & \\
& c &
\end{tikzcd}
$$

A key observation

From this follows that there exists a cofinal functor

$$\mathbf{Elts}\,(G) \times \mathbf{Elts}\,(F) \longrightarrow \mathbf{Elts}\,(G \mathbin{\hat{\otimes}} F)$$

making the diagram commute:

$$\begin{tikzcd}
\mathbf{Elts}\,(G) \times \mathbf{Elts}\,(F) \arrow[rr, "\text{cofinal}"] \arrow[dr, swap, "\otimes \circ (\pi_G \times \pi_F)"] & & \mathbf{Elts}\,(G \mathbin{\hat{\otimes}} F) \arrow[dl, "\pi_{G \hat{\otimes} F}"] \\
& \mathbf{C} &
\end{tikzcd}$$

in the category $\mathbf{Cat}$ of categories and functors.

A key observation

The category $\mathbf{Cat}/\mathbf{C}$ inherits a tensor product

$$\tilde{\otimes} \quad : \quad \mathbf{Cat}/\mathbf{C} \times \mathbf{Cat}/\mathbf{C} \longrightarrow \mathbf{Cat}/\mathbf{C}$$

from the monoidal structure of the category $\mathbf{C}$.

The Day tensor product

$$\hat{\otimes} \quad : \quad \mathbf{DiscOpFib}/\mathbf{C} \times \mathbf{DiscOpFib}/\mathbf{C} \longrightarrow \mathbf{DiscOpFib}/\mathbf{C}$$

is the monoidal structure obtained by transporting $\tilde{\otimes}$ along the adjunction

$$\begin{tikzcd} \mathbf{Cat}/\mathbf{C} \arrow[rr, bend left, "Free"] \arrow[rr, phantom, "\perp"] & & \mathbf{DiscOpFib}/\mathbf{C} \arrow[ll, bend left, "Inclusion"] \end{tikzcd}$$

The convolution product on double categories

Extending the Day construction

The convolution product on double categories

Given two covariant presheaves

$$F, G \quad : \quad \mathbb{D}_1 \longrightarrow \mathbf{Set}$$

on a double category $\mathbb{D}$ with horizontal composition

$$\diamond_h \quad : \quad \mathbb{D}_2 = \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \longrightarrow \mathbb{D}_1$$

the **convolution product** of F and G is the covariant presheaf

$$G * F \quad : \quad \mathbb{D}_1 \longrightarrow \mathbf{Set}$$

defined by the coend formula:

$$G * F \;=\; t \;\mapsto\; \int^{(s,r)\,\in\,\mathbb{D}_2} \mathbb{D}_1(s \diamond_h r, t) \times G(s) \times F(r)$$

The convolution product

Equivalently, the convolution product

$$G * F \quad : \quad \mathbb{D}_1 \longrightarrow \mathbf{Set}$$

may be defined as the **left Kan extension** of the functor

$$\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \xrightarrow{\ proj\ } \mathbb{D}_1 \times \mathbb{D}_1 \xrightarrow{\ G \times F\ } \mathbf{Set} \times \mathbf{Set} \xrightarrow{\ \times\ } \mathbf{Set}$$

along the tensor product functor:

$$\begin{tikzcd}
\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[r, "proj"] \arrow[drr, "\diamond_h"'] & \mathbb{D}_1 \times \mathbb{D}_1 \arrow[rr, "G \times F"] & {} \arrow[d, phantom, "\Downarrow"] & \mathbf{Set} \times \mathbf{Set} \arrow[r, "\times"] & \mathbf{Set} \\
 & & \mathbb{D}_1 \arrow[urr, "\psi * \varphi"'] & &
\end{tikzcd}$$

What does the coend formula mean?

An element of the coend

$$G * F\,(t) \;=\; \int^{(s,r)\,\in\,\mathbb{D}_2} \mathbb{D}_1(s \diamond_h r, t) \times G(s) \times F(r)$$

consists of a double cell of the form

$$\begin{tikzcd} B & C \arrow[l, "s"'] \arrow[d, Rightarrow, "\gamma"] & A \arrow[l, "r"'] \arrow[d, "f"] \\ B' \arrow[from=1-1, "g"'] & & A' \arrow[ll, "t"] \end{tikzcd}$$

together with a pair of elements

$$y \in G(s) \qquad\qquad x \in F(r)$$

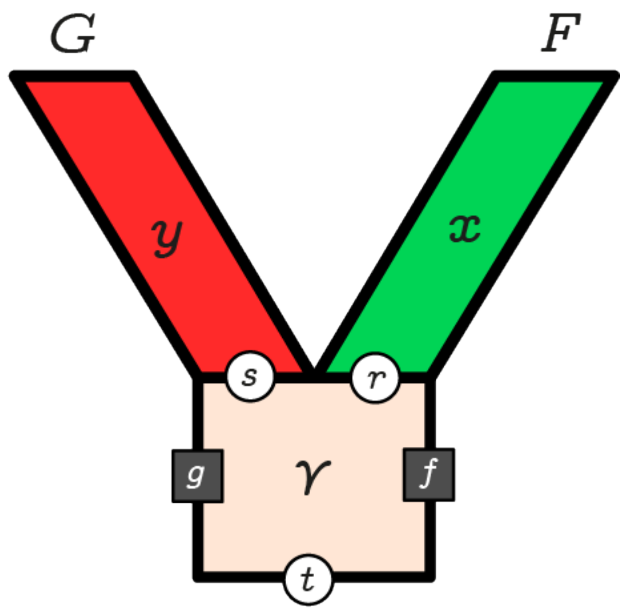
considered modulo an equivalence relation noted $\sim$.

What does the coend formula mean?

We find enlightening to draw the triple

$$\Big(\quad s \diamond_h r \xRightarrow{\gamma} t \quad , \quad x \in F(r) \quad , \quad y \in G(s) \quad \Big)$$

in the following way:



This picture is the reason we like to speak of the rabbit calculus.

What does the coend formula mean?

Suppose given a pair of elements

$$x \in F(r) \qquad\qquad y \in G(s)$$

a pair of double cells

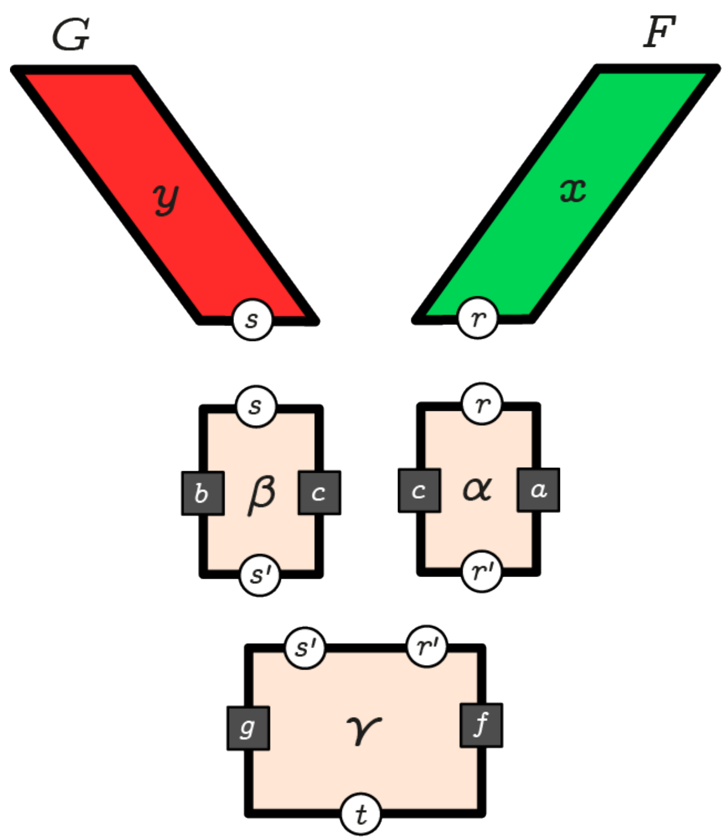
$$\alpha \ : \ r \Longrightarrow r' \qquad\qquad \beta \ : \ s \Longrightarrow s'$$

and a double cell

$$\gamma \ : \ s' \diamond_h r' \Longrightarrow t$$

What does the coend formula mean?

The five components may be depicted as follows:

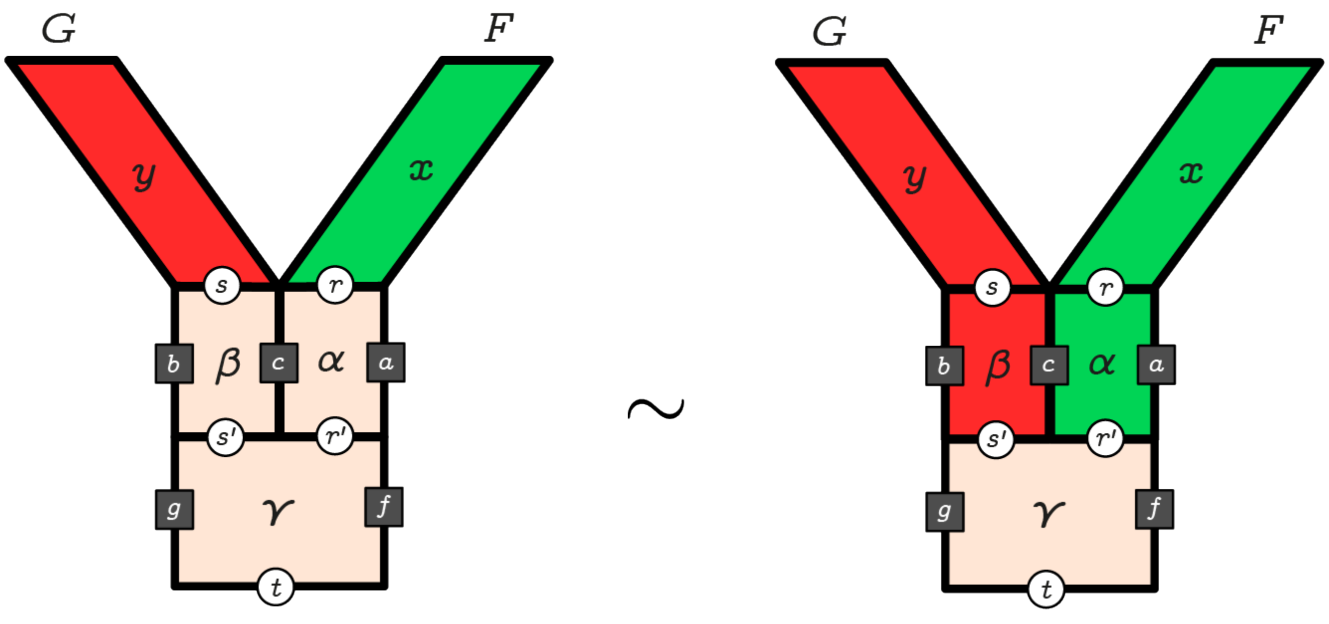


What does the coend formula mean?

The equivalence relation $\sim$ defined by the coend

$$G * F\ (t) \ = \int^{(s,r)\in\mathbb{D}_2} \mathbb{D}_1(s \diamond_h r, t) \times G(s) \times F(r)$$

identifies every triple of the form



Main structural theorem

Theorem [Behr, PAM, Zeilberger]

The convolution product

$$G, F \quad \mapsto \quad G * F$$

on a double category $\mathbb{D}$ defines a functor

$$* \quad : \quad \widehat{\mathbb{D}} \times \widehat{\mathbb{D}} \longrightarrow \widehat{\mathbb{D}}$$

which equips the category of covariant presheaves

$$\widehat{\mathbb{D}} \quad := \quad [\mathbb{D}_1, \mathbf{Set}]$$

with the structure of an **oplax monoidal closed category**.

What oplax monoidal means...

The category of covariant presheaves

$$\widehat{\mathbb{D}} \quad := \quad [\mathbb{D}_1, \mathbf{Set}]$$

comes equipped with a **family of convolution products**

$$*_n \quad : \quad \widehat{\mathbb{D}} \times \cdots \times \widehat{\mathbb{D}} \longrightarrow \widehat{\mathbb{D}}$$

where we use the notation

$$(F_n * \cdots * F_1) \quad := \quad *_n \, (F_n, \ldots, F_1)$$

for the n-ary product of n covariant presheaves

$$F_n, \ldots, F_1 \quad : \quad \mathbb{D}_1 \longrightarrow \mathbf{Set}.$$

The ternary convolution product

Typically, the ternary convolution product

$$H * G * F \quad : \quad \mathbf{C} \longrightarrow \mathbf{Set}$$

of three covariant presheaves H, G, F is defined by the coend formula

$$H * G * F \;=\; u \;\mapsto\; \int^{(t,s,r) \in \mathbb{D}_3} \mathbb{D}_1(t \diamond_h s \diamond_h r, u) \times H(t) \times G(s) \times F(r)$$

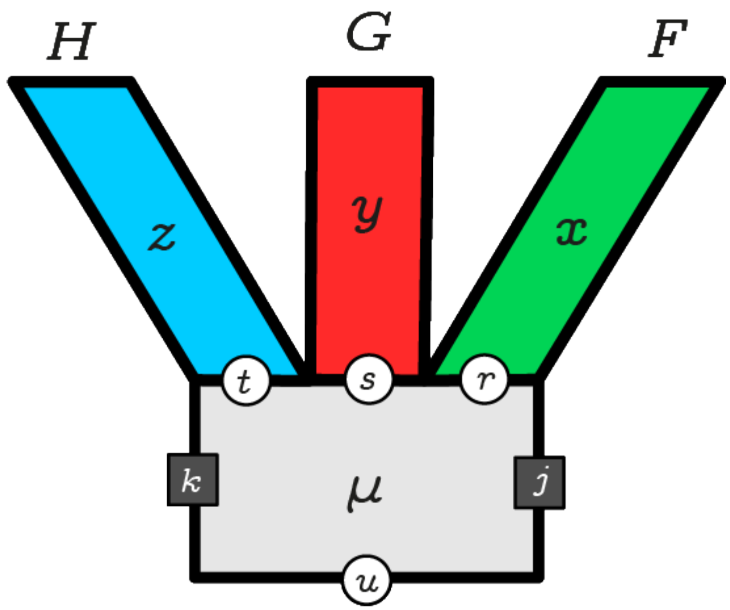
where $\mathbb{D}_3$ is the category of horizontal paths of length 3.

The ternary convolution product

The elements of the ternary convolution product are quadruples

$$\left(\quad t \diamond_h s \diamond_h r \overset{\delta}{\Longrightarrow} u \quad , \quad x \in F(r) \quad , \quad y \in G(s) \quad , \quad z \in G(t) \quad \right)$$

which may be depicted in the following way:

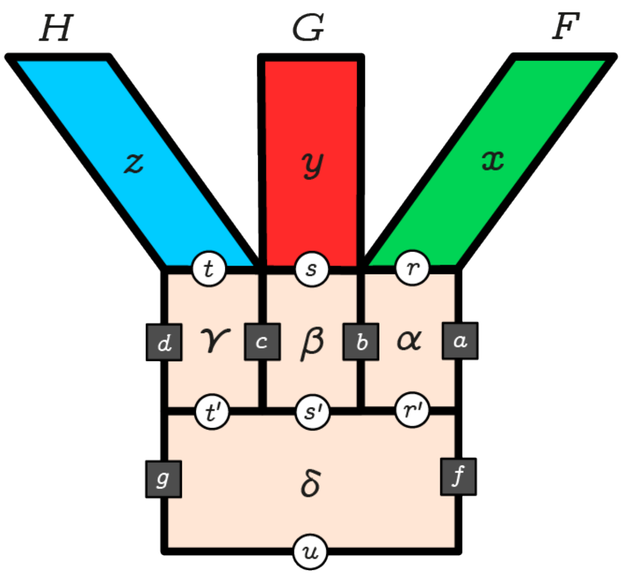


The ternary convolution product

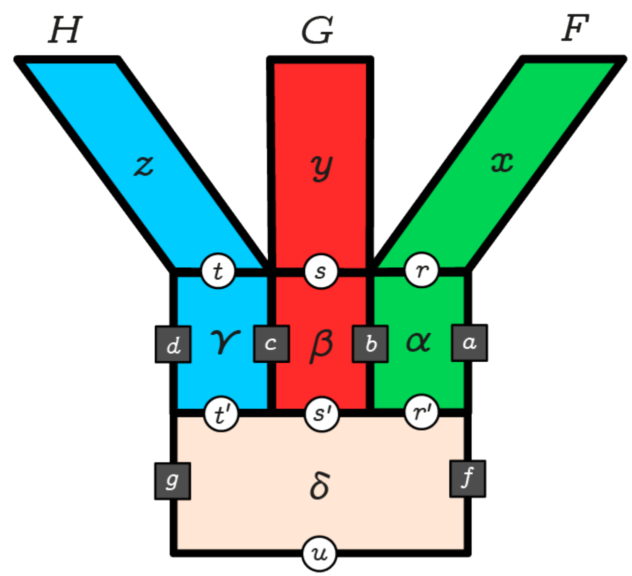
The elements of the convolution product

$$\left(\quad t \diamond_h s \diamond_h r \overset{\delta}{\Longrightarrow} u \quad , \quad x \in F(r) \quad , \quad y \in G(s) \quad , \quad z \in G(t) \quad \right)$$

are identified modulo the equivalence relation:



$\sim$

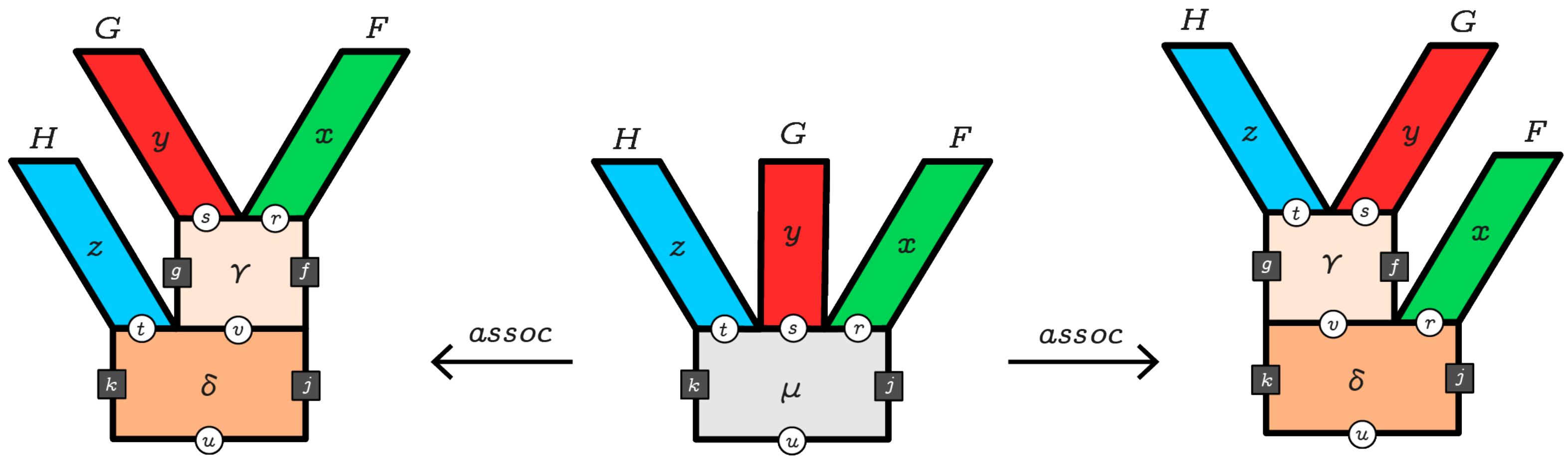


What oplax monoidal means...

The convolution products are related by associativity maps such as

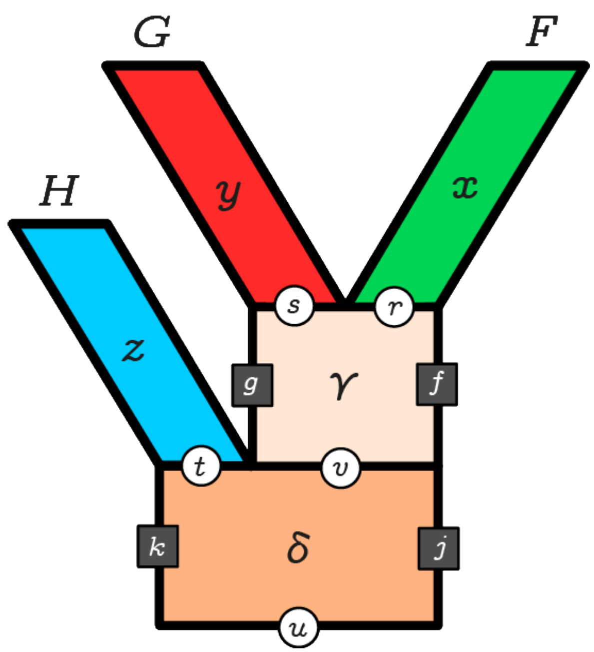
$$H * (G * F) \xleftarrow{\;assoc\;} (H * G * F) \xrightarrow{\;assoc\;} (H * G) * F$$

which are **not reversible** in general, for the following reason:



What oplax monoidal means...

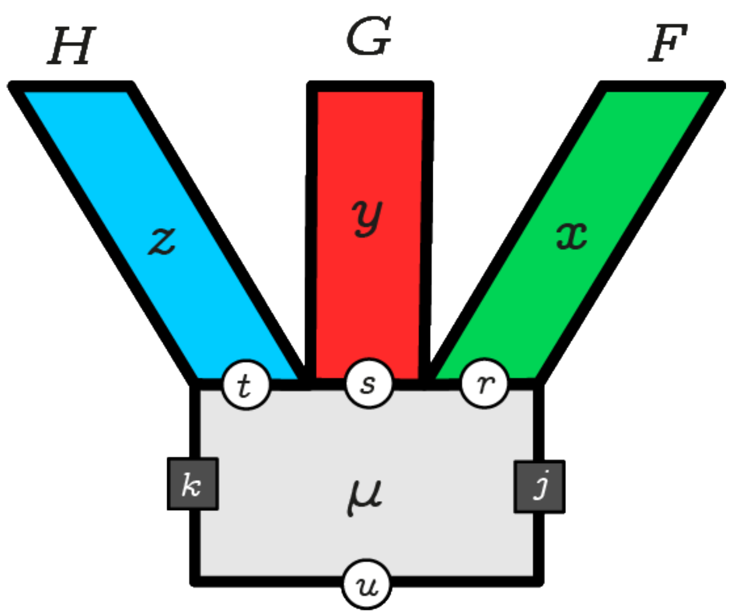
In a general double category $\mathbb{D}$, not every composite shape of the form



defining an element of the presheaf $H * (G * F)$ at instance $u : A \longrightarrow A'$

What oplax monoidal means...

is equivalent modulo $\sim$ in $\mathbb{D}$ to a ternary shape of the form



defining an element of $H * G * F$ at the same instance $u : A \longrightarrow A'$.

Sketch of the proof

Key observation. The projection functor

$$\pi_{G*F} \quad : \quad \mathbf{Elts}\,(G * F) \longrightarrow \mathbb{D}_1$$

associated to the binary convolution product

$$G * F \quad : \quad \mathbb{D}_1 \longrightarrow \mathbf{Set}$$

is the **free discrete opfibration** associated to the functor

$$\mathbf{Elts}\,(G) \circledast \mathbf{Elts}\,(F) = \mathbf{Elts}\,(G) \times_{\mathbb{D}_0} \mathbf{Elts}\,(F) \xrightarrow{\pi_G \circledast \pi_F} \mathbb{D}_2 \xrightarrow{h_2} \mathbb{D}_1$$

obtained by composing the two projection functors above $\mathbb{D}_1$

$$\mathbf{Elts}\,(G) \xrightarrow{\pi_G} \mathbb{D}_1 \qquad\qquad \mathbf{Elts}\,(F) \xrightarrow{\pi_F} \mathbb{D}_1$$

Sketch of the proof

$$
\begin{tikzcd}
 & \mathbf{Elts}\,(G) \arrow[d, "\pi_G"'] & & \mathbf{Elts}\,(F) \arrow[d, "\pi_F"] & \\
 & \mathbb{D}_1 \arrow[dl, "T"'] \arrow[dr, "S"] & & \mathbb{D}_1 \arrow[dl, "T"'] \arrow[dr, "S"] & \\
\mathbb{D}_0 & & \mathbb{D}_0 & & \mathbb{D}_0
\end{tikzcd}
$$

Sketch of the proof

$$
\begin{tikzcd}
 & & \mathbf{Elts}\,(G) \circledast \mathbf{Elts}\,(F) \arrow[dll] \arrow[drr] & & \\
 \mathbf{Elts}\,(G) \arrow[d, "\pi_G"'] & & & & \mathbf{Elts}\,(F) \arrow[d, "\pi_F"] \\
 \mathbb{D}_1 \arrow[d, "T"'] \arrow[drr, "S"] & & & & \mathbb{D}_1 \arrow[dll, "T"'] \arrow[d, "S"] \\
 \mathbb{D}_0 & & \mathbb{D}_0 & & \mathbb{D}_0
\end{tikzcd}
$$

Sketch of the proof

$$
\begin{tikzcd}
 & & \mathbf{Elts}\,(G) \circledast \mathbf{Elts}\,(F) \arrow[dl] \arrow[dr] \arrow[dd, dashed, "\pi_G \circledast \pi_F"] & & \\
 & \mathbf{Elts}\,(G) \arrow[dd, "\pi_G"'] & & \mathbf{Elts}\,(F) \arrow[dd, "\pi_F"] & \\
 & & \mathbb{D}_2 \arrow[dl, "T"'] \arrow[dr, "S"] & & \\
 & \mathbb{D}_1 \arrow[dl, "T"'] \arrow[dr, "S"] & & \mathbb{D}_1 \arrow[dl, "T"'] \arrow[dr, "S"] & \\
\mathbb{D}_0 & & \mathbb{D}_0 & & \mathbb{D}_0
\end{tikzcd}
$$

Sketch of the proof

$$\begin{tikzcd}
 & & \mathbf{Elts}\,(G) \circledast \mathbf{Elts}\,(F) \arrow[dl] \arrow[dr] \arrow[dd, dashed, "\pi_G \circledast \pi_F"] & & \\
 & \mathbf{Elts}\,(G) \arrow[dd, "\pi_G"'] & & \mathbf{Elts}\,(F) \arrow[dd, "\pi_F"] & \\
 & & \mathbb{D}_2 \arrow[dl, "T"'] \arrow[dr, "S"] \arrow[dd, "h_2"] & & \\
 & \mathbb{D}_1 \arrow[dl, "T"'] & & \mathbb{D}_1 \arrow[dr, "S"] & \\
\mathbb{D}_0 & & \mathbb{D}_1 \arrow[ll, "T"] \arrow[rr, "S"'] & & \mathbb{D}_0
\end{tikzcd}$$

Sketch of the proof

Similarly, the projection functor

$$\pi_{H*G*F} \quad : \quad \mathbf{Elts}\,(H * G * F) \longrightarrow \mathbb{D}_1$$

associated to the ternary convolution product

$$H * G * F \quad : \quad \mathbb{D}_1 \longrightarrow \mathbf{Set}$$

is the **free discrete opfibration** associated to the functor

$$\mathbf{Elts}\,(H) \circledast \mathbf{Elts}\,(G) \circledast \mathbf{Elts}\,(F) \xrightarrow{\pi_H \circledast \pi_G \circledast \pi_F} \mathbb{D}_3 \xrightarrow{h_3} \mathbb{D}_1$$

obtained by composing the three projection functors above $\mathbb{D}_1$

$$\mathbf{Elts}\,(H) \xrightarrow{\pi_H} \mathbb{D}_1 \qquad \mathbf{Elts}\,(G) \xrightarrow{\pi_G} \mathbb{D}_1 \qquad \mathbf{Elts}\,(F) \xrightarrow{\pi_F} \mathbb{D}_1$$

Sketch of the proof

$$\begin{tikzcd}[column sep=small]
 & \mathbf{Elts}\,(H) \arrow[d, "\pi_H"'] & & \mathbf{Elts}\,(G) \arrow[d, "\pi_G"] & & \mathbf{Elts}\,(F) \arrow[d, "\pi_F"] & \\
 & \mathbb{D}_1 \arrow[dl, "T"'] \arrow[dr, "S"] & & \mathbb{D}_1 \arrow[dl, "T"'] \arrow[dr, "S"] & & \mathbb{D}_1 \arrow[dl, "T"'] \arrow[dr, "S"] & \\
\mathbb{D}_0 & & \mathbb{D}_0 & & \mathbb{D}_0 & & \mathbb{D}_0
\end{tikzcd}$$

Sketch of the proof

$$
\begin{tikzcd}
 & & & \mathbf{Elts}(H) \circledast \mathbf{Elts}(G) \circledast \mathbf{Elts}(F) \arrow[dll] \arrow[d] \arrow[drr] & & & \\
 & \mathbf{Elts}(H) \arrow[d, "\pi_H"'] & & \mathbf{Elts}(G) \arrow[d, "\pi_G"] & & \mathbf{Elts}(F) \arrow[d, "\pi_F"] & \\
 & \mathbb{D}_1 \arrow[dl, "T"'] \arrow[dr, "S"] & & \mathbb{D}_1 \arrow[dl, "T"'] \arrow[dr, "S"] & & \mathbb{D}_1 \arrow[dl, "T"'] \arrow[dr, "S"] & \\
\mathbb{D}_0 & & \mathbb{D}_0 & & \mathbb{D}_0 & & \mathbb{D}_0
\end{tikzcd}
$$

Sketch of the proof

$$\begin{tikzcd}
 & & & \mathbf{Elts}\,(H) \circledast \mathbf{Elts}\,(G) \circledast \mathbf{Elts}\,(F) \arrow[d, dashed, "\pi_H \circledast \pi_G \circledast \pi_F"] & & & \\
 & & & \mathbb{D}_3 \arrow[dll] \arrow[d] \arrow[drr] & & & \\
 & \mathbb{D}_1 \arrow[dl, "T"'] \arrow[dr, "S"] & & \mathbb{D}_1 \arrow[dl, "T"'] \arrow[dr, "S"] & & \mathbb{D}_1 \arrow[dl, "T"'] \arrow[dr, "S"] & \\
\mathbb{D}_0 & & \mathbb{D}_0 & & \mathbb{D}_0 & & \mathbb{D}_0
\end{tikzcd}$$

Sketch of the proof

$$
\begin{tikzcd}
& & \mathbf{Elts}\,(H) \circledast \mathbf{Elts}\,(G) \circledast \mathbf{Elts}\,(F) \arrow[d, dashed, "\pi_H \circledast \pi_G \pi_F"] & & \\
& & \mathbb{D}_3 \arrow[dl] \arrow[dr] \arrow[dd, "h_3"] & & \\
& \mathbb{D}_1 \arrow[dl, "T"'] & & \mathbb{D}_1 \arrow[dr, "S"] & \\
\mathbb{D}_0 & & \mathbb{D}_1 \arrow[ll, "T"] \arrow[rr, "S"'] & & \mathbb{D}_0
\end{tikzcd}
$$

Main argument of the proof

The category $\mathbf{Cat}/\mathbb{D}_1$ inherits a monoidal structure

$$\circledast \quad : \quad \mathbf{Cat}/\mathbb{D}_1 \times \mathbf{Cat}/\mathbb{D}_1 \longrightarrow \mathbf{Cat}/\mathbb{D}_1$$

computed by pullback using the double categorical structure of $\mathbb{D}$.

The convolution product

$$* \quad : \quad \mathbf{DiscOpFib}/\mathbb{D}_1 \times \mathbf{DiscOpFib}/\mathbb{D}_1 \longrightarrow \mathbf{DiscOpFib}/\mathbb{D}_1$$

is the **oplax monoidal structure** obtained by transporting on $\widehat{\mathbb{D}} = [\mathbb{D}_1, \mathbf{Set}]$ the strong monoidal structure $\circledast$ on $\mathbf{Cat}/\mathbb{D}_1$ along the adjunction

$$\begin{tikzcd} \mathbf{Cat}/\mathbb{D}_1 \arrow[rr, bend left, "Free"] \arrow[rr, phantom, "\perp"] && \mathbf{DiscOpFib}/\mathbb{D}_1 \arrow[ll, bend left, "Inclusion"] \end{tikzcd}$$

Cylindrical decomposition property

A sufficient condition to ensure strong associativity

Towards strong associativity

We want to find a **sufficient condition** on a double category

$$(\,\mathbb{D}\,,\,h_n : \mathbb{D}_n \longrightarrow \mathbb{D}_1\,)$$

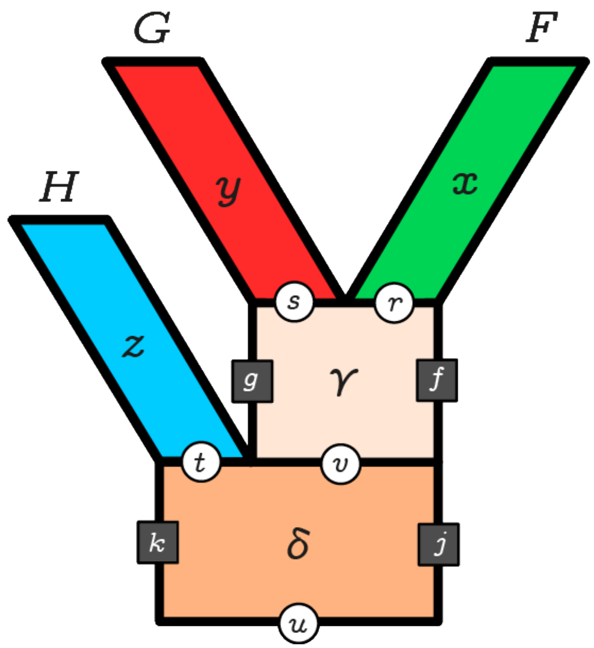
ensuring that the **associativity maps** of the convolution product

$$H * (G * F) \xleftarrow{assoc} (H * G * F) \xrightarrow{assoc} (H * G) * F$$

are **reversible**.

Towards strong associativity

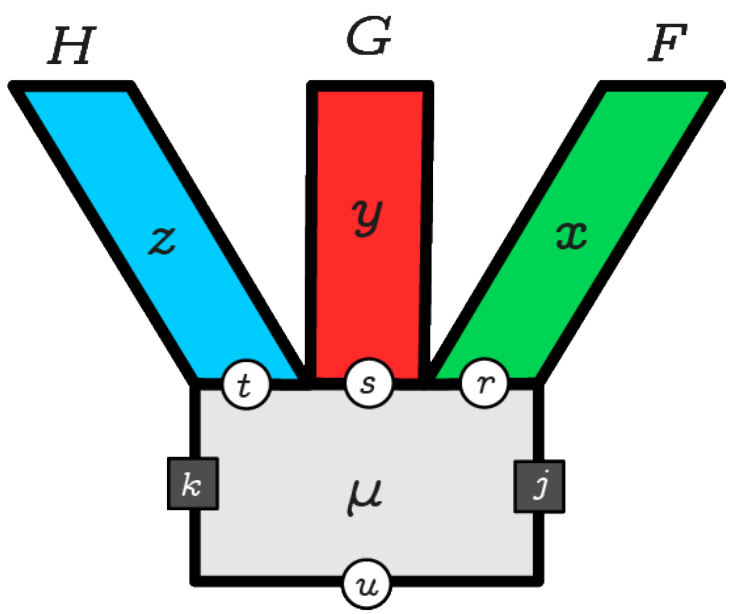
In particular, this requires to show that every **composite shape**



defining an element of the presheaf $H * (G * F)$ at instance $u : A \longrightarrow A'$

Towards strong associativity

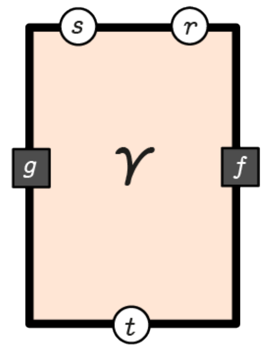
is equivalent modulo $\sim$ in $\mathbb{D}$ to a ternary shape of the form



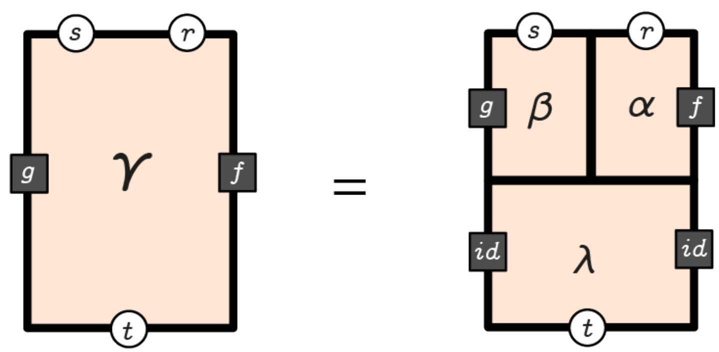
defining an element of $H * G * F$ at the same instance $u : A \longrightarrow A'$.

Towards strong associativity

Suppose that every double cell of the form

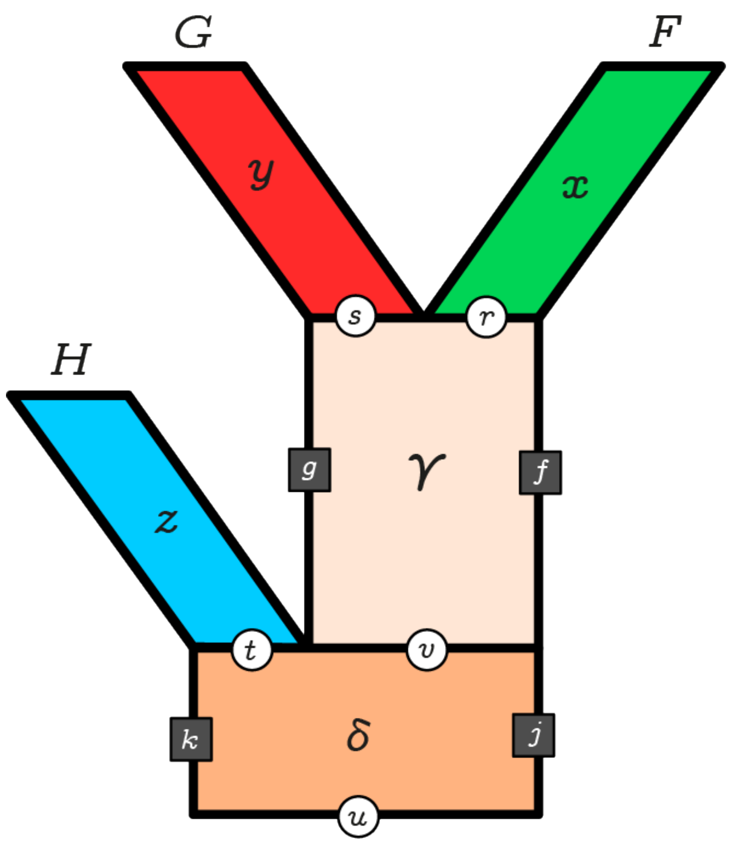


factors in the following way:



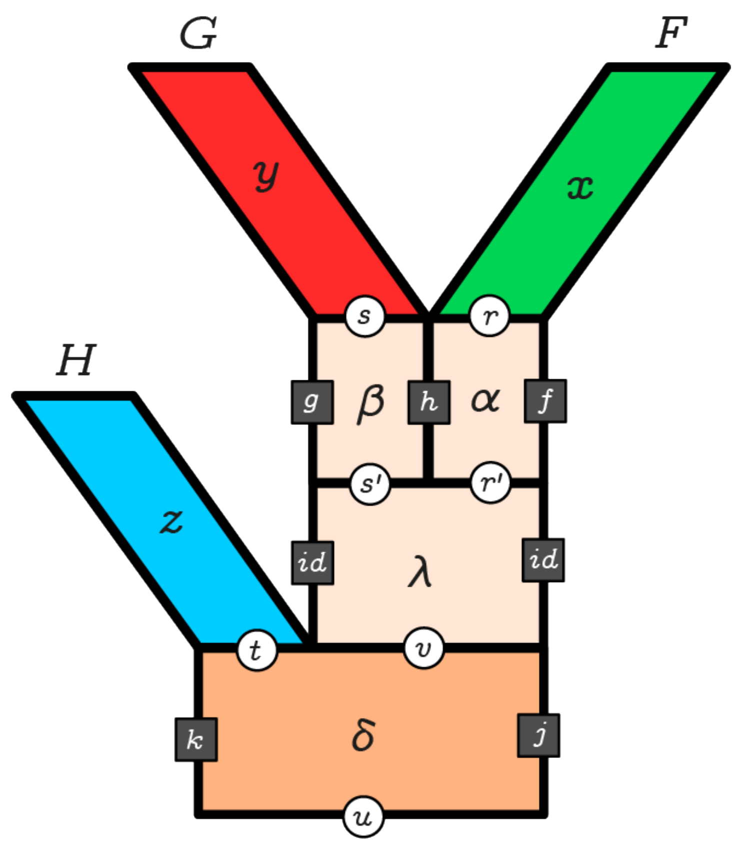
Towards strong associativity

In that case, one can rewrite the original composite shape



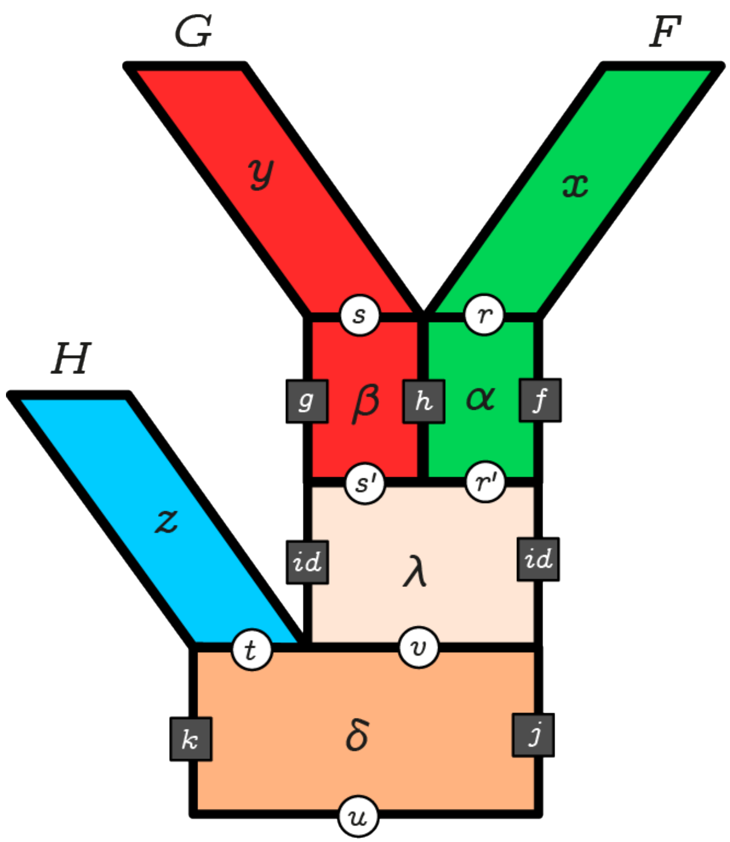
Towards strong associativity

We then into the shape where the cell γ has been factored:



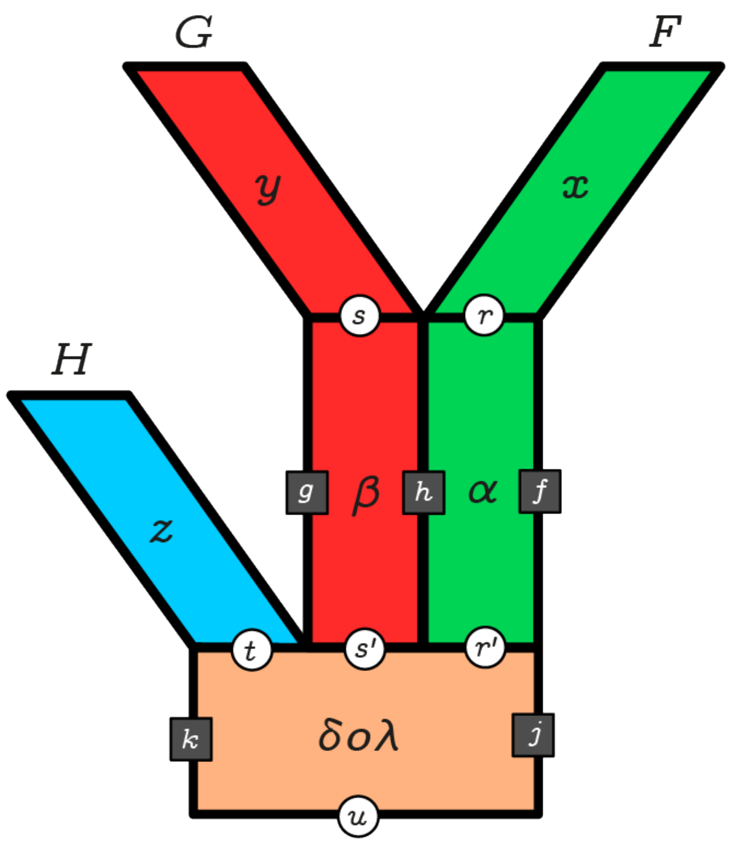
Towards strong associativity

then into the equivalent shape using the equivalence relation $\sim$



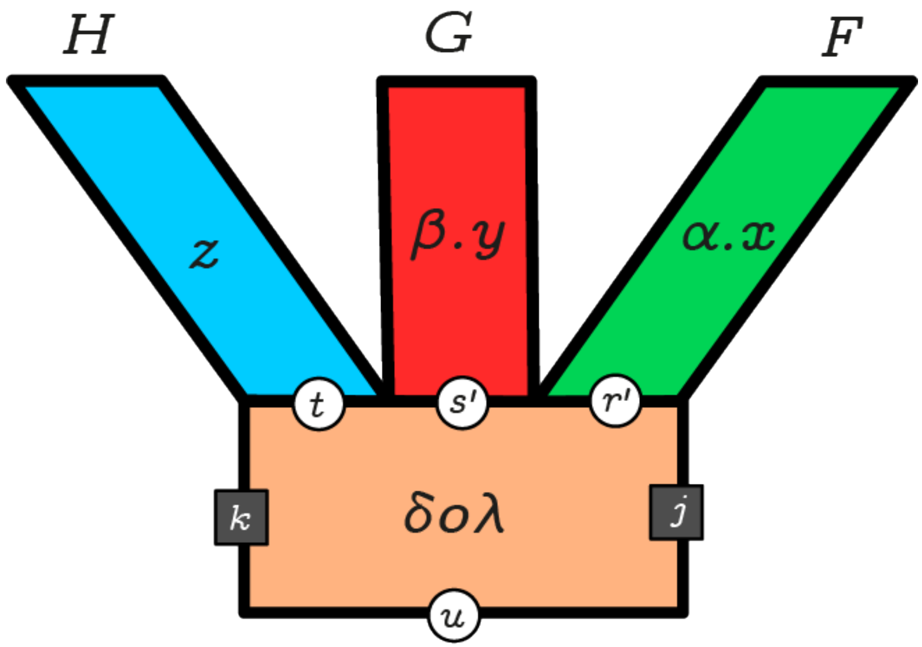
Towards strong associativity

then into the equal shape by vertical composition:



Towards strong associativity

and finally in the ternary shape we were looking for:



The cylinder categories

Every double category $\mathbb{D}$ comes equipped with a family of categories

$$\mathbf{Cyl}_{\mathbb{D}}\,[n]$$

called **cylinder categories** and defined in the following way:

- ▷ the objects of $\mathbf{Cyl}_{\mathbb{D}}\,[n]$ are the tuples

$$\sigma \quad = \quad (\,s_n\,,\,\ldots\,,\,s_1\,,\,s\,,\,\sigma : s_n \diamond_h \cdots \diamond_h s_1 \Rightarrow s\,)$$

  defining a **globular cell** of the form

$$\begin{tikzcd}
A_n \arrow[d, "id"'] & A_{n-1} \arrow[l, "s_n"'] & \cdots & A_4 & A_3 \arrow[l, "s_3"'] \arrow[d, phantom, "\Downarrow\sigma"] & A_2 \arrow[l, "s_2"'] & A_1 \arrow[l, "s_1"'] \arrow[d, "id"] \\
A_n & & & & & & A_1 \arrow[llllll, "s"]
\end{tikzcd}$$

The cylinder categories

- given globular cells

$$\sigma \quad = \quad (\, s_n\,,\,\ldots\,,\,s_1\,,\,s\,,\,\sigma : s_n \diamond_h \cdots \diamond_h s_1 \;\Rightarrow\; s\,)$$

$$\tau \quad = \quad (\, t_n\,,\,\ldots\,,\,t_1\,,\,t\,,\,\tau : t_n \diamond_h \cdots \diamond_h t_1 \;\Rightarrow\; t\,)$$

the morphisms of $\mathbf{Cyl}_{\mathbb{D}}[n]$ of the form

$$(\varphi_n, \cdots, \varphi_1, \varphi) \quad : \quad \sigma \longrightarrow \tau$$

are tuples consisting of a map in $\mathbb{D}_n$

$$(\varphi_n, \ldots, \varphi_1) \quad : \quad (\, s_n\,,\,\ldots\,,\,s_1\,) \Rightarrow (\, t_n\,,\,\ldots\,,\,t_1\,)$$

and of a double cell

$$\varphi \quad : \quad s \Rightarrow t$$

The cylinder categories

such that the double cell $\varphi \circ \sigma$ depicted below

$$
\begin{tikzcd}
A_n \arrow[d, "id"'] & A_{n-1} \arrow[l, "s_n"'] & \cdots & A_3 \arrow[dr, phantom, "\Downarrow\sigma"] & A_2 \arrow[l, "s_3"'] & A_1 \arrow[l, "s_2"'] & A_0 \arrow[l, "s_1"'] \arrow[d, "id"] \\
A_n \arrow[d, "a_n"'] & & & \arrow[dr, phantom, "\Downarrow\varphi"] & & & A_0 \arrow[llllll, "s"] \arrow[d, "a_0"] \\
B_n & & & & & & B_0 \arrow[llllll, "t"]
\end{tikzcd}
$$

The cylinder categories

is equal to the double cell $\tau \circ (\varphi_n \diamond_h \ldots \diamond_h \varphi_1)$ depicted below

$$\begin{tikzcd}
A_n \arrow[d, "a_n"'] & A_{n-1} \arrow[l, "s_n"'] \arrow[d, "a_{n-1}" description] \arrow[dl, phantom, "\Downarrow \varphi_n"] & \cdots & A_3 \arrow[d, "a_3" description] & A_2 \arrow[l, "s_3"'] \arrow[d, "a_2" description] \arrow[dl, phantom, "\Downarrow \varphi_3"] & A_1 \arrow[l, "s_2"'] \arrow[d, "a_1" description] \arrow[dl, phantom, "\Downarrow \varphi_2"] & A_0 \arrow[l, "s_1"'] \arrow[d, "a_0"] \arrow[dl, phantom, "\Downarrow \varphi_1"] \\
B_n \arrow[d, "id"'] & B_{n-1} \arrow[l, "t_n" description] & \cdots & B_3 \arrow[d, phantom, "\Downarrow \tau"] & B_2 \arrow[l, "t_3" description] & B_1 \arrow[l, "t_2" description] & B_0 \arrow[l, "t_1" description] \arrow[d, "id"] \\
B_n & & & & & & B_0 \arrow[llllll, "t"]
\end{tikzcd}$$

The cylindrical decomposition property

Key observation: each composition functor

$$h_n \quad : \quad \mathbb{D}_n \longrightarrow \mathbb{D}_1$$

of the double category $\mathbb{D}$ factors as

$$\mathbb{D}_n \longrightarrow \mathbf{Cyl}_{\mathbb{D}}[n] \xrightarrow{\pi_n} \mathbb{D}_1$$

Definition. A double category $\mathbb{D}$ satisfies

the n-cylindrical decomposition property (n-CDP)

when the functor

$$\mathbf{Cyl}_{\mathbb{D}}[n] \xrightarrow{\pi_n} \mathbb{D}_1$$

is an opfibration.

Main theorem

Theorem. [Behr,PAM,Zeilberger]

Suppose that a double category $\mathbb{D}$ satisfies

the n-cylindrical decomposition property (n-CDP)

for all $n \in \mathbb{N}$.

In that case, the convolution product defines a functor

$$* \quad : \quad \widehat{\mathbb{D}} \times \widehat{\mathbb{D}} \longrightarrow \widehat{\mathbb{D}}$$

which equips the category of covariant presheaves

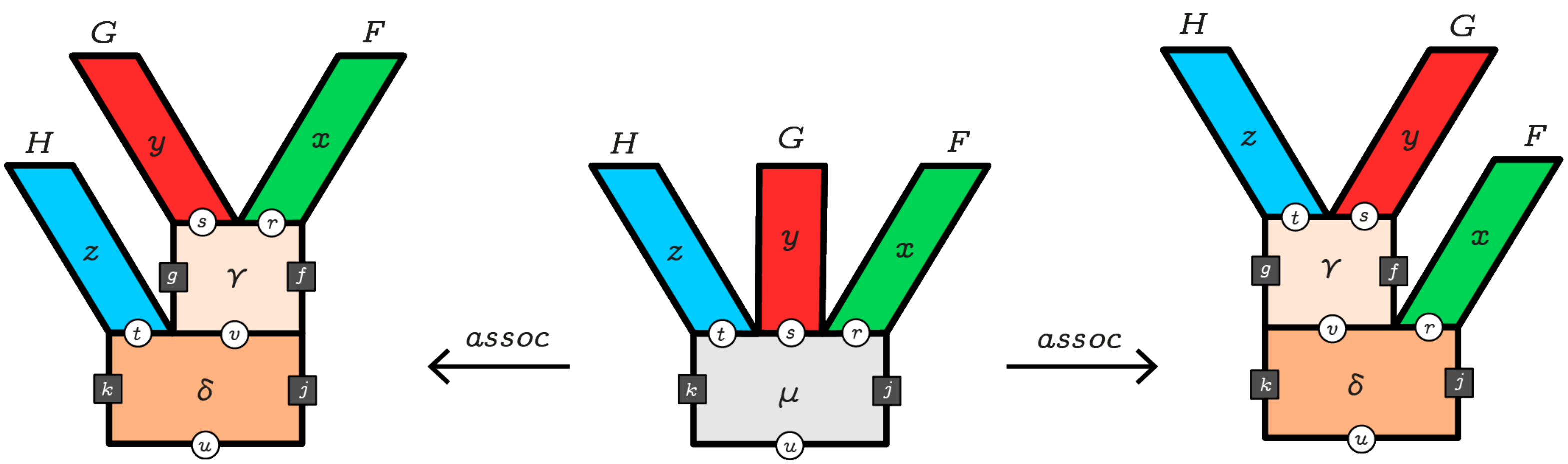
$$\widehat{\mathbb{D}} \quad := \quad [\mathbb{D}_1, \mathbf{Set}]$$

with the structure of an **strong monoidal closed category**.

Main theorem

In particular, the associativity maps are **reversible** in that case:

$$H * (G * F) \xleftarrow{\;assoc\;} (H * G * F) \xrightarrow{\;assoc\;} (H * G) * F$$



Reversibility comes from the cylindrical decomposition property of $\mathbb{D}$.

Illlustrations

The theorem applies to the following situations:

- ▷ every **bicategory** $\mathbb{D} = \mathcal{W}$ satisfies n-CDP,
- ▷ every **framed bicategory** $\mathbb{D} = \mathcal{W}$ satisfies n-CDP for $n \geq 1$,
- ▷ the double category $\mathbb{D} = \mathbf{DPO}$ satisfies n-CDP for $n \geq 1$.

More generally, the theorem enables us to use the convolution product for a number of categorical graph rewriting frameworks.

Categorifying rule algebras

Composing representable presheaves by convolution

Categorification of rule algebras

One main ingredient of **rule algebras** is the following equation

$$\delta(r) \star \delta(s) \quad = \quad \sum_{\mu \in \mathcal{M}_r(s)} \delta(r_\mu s)$$

where

- $\triangleright$ $\mathcal{M}_r(s)$ is the set of **admissible matches** of rule r into rule s
- $\triangleright$ $r_\mu s$ denotes one possible way to get a **composite rule** from r and s.

Similarly, we want to find sufficient conditions on $\mathbb{D}$ such that

$$\hat{\Delta}_r * \hat{\Delta}_s \quad = \quad \sum_{\mu \in \mathcal{M}_r(s)} \hat{\Delta}_{r_\mu s)}$$

where the sum is now **set-theoretic union**.

Multi-sums

Suppose that A and B are objects in a category $\mathbf{C}$.

Definition. A **multi-sum** of A and B is a family of cospans

$$(A \xrightarrow{a_i} U_i \xleftarrow{b_i} B)_{i \in I}$$

such that for any cospan

$$A \xrightarrow{f} X \xleftarrow{g} B$$

there exists a unique $i \in I$ and a unique morphism

$$[f, g] \quad : \quad U_i \xrightarrow{f} X$$

such that

$$f = [f, g] \circ a_i \quad \text{and} \quad g = [f, g] \circ b_i.$$

Categorification of rule algebras

Assume $\mathbb{D}$ is a small double category satisfying

- ▷ the vertical category $\mathbb{D}_0$ has multi-sums,
- ▷ the source and target functors $S, T : \mathbb{D}_1 \to \mathbb{D}_0$ are opfibrations.

In that case, the convolution product of two representable presheaves is isomorphic to the sum of representables

$$\hat{\Delta}_{r_2} * \hat{\Delta}_{r_1} \quad \cong \quad \sum_{i \in I} \quad \hat{\Delta}_{r_2 \langle c_i \rangle \diamond_h \langle b_i \rangle r_1}$$

where the multi-sum of B and C is given by a family of cospans

$$(\ B \xrightarrow{b_i} U_i \xleftarrow{c_i} C \ )_{i \in I}$$

and where $r_2\langle c_i \rangle$ denotes the S-pushforward of r_2 along c_i
and $\langle b_i \rangle r_1$ denotes the T-pushforward of r_1 along b_i.

Thank you!

$$
\begin{tikzcd}
D \arrow[d] & C \arrow[l, "r_2"'] \arrow[dr, "g"] \arrow[dl, phantom, "\Downarrow\alpha_2"] & & B \arrow[dl, "f"'] \arrow[dr, phantom, "\Downarrow\alpha_1"] & A \arrow[l, "r_1"'] \arrow[d] \\
\cdot \arrow[d] & & X \arrow[ll, "s_2"] \arrow[dd, phantom, "\Downarrow\beta"] & & \cdot \arrow[ll, "s_1"] \arrow[d] \\
\cdot & & & & \cdot \arrow[llll, "r"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
D \arrow[d] & {} \arrow[d, phantom, "\Downarrow \alpha_2"] & C \arrow[ll, "r_2"'] \arrow[dr, "g"] & & B \arrow[dl, "f"'] & {} \arrow[d, phantom, "\Downarrow \alpha_1"] & A \arrow[d] \arrow[ll, "r_1"'] \\
\cdot & {} & & X \arrow[lll, "s_2"] & & {} & \cdot \arrow[lll, "s_1"]
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
D \arrow[d] & {} \arrow[d, phantom, "\Downarrow S\text{-}opcart"] & C \arrow[ll, "r_2"'] \arrow[dr, "c_i"] & & B \arrow[dl, "b_i"'] & {} \arrow[d, phantom, "\Downarrow T\text{-}opcart"] & A \arrow[d] \arrow[ll, "r_1"'] \\
\cdot \arrow[d] & {} \arrow[d, phantom, "\Downarrow \tilde{\alpha}_2"] & & U_i \arrow[lll, "r_2\langle c_i\rangle"] \arrow[d, "{[f,g]}"] & & {} \arrow[d, phantom, "\Downarrow \tilde{\alpha}_1"] & \cdot \arrow[d] \arrow[lll, "\langle b_i\rangle r_1"'] \\
\cdot & {} & & X \arrow[lll, "s_2"] & & {} & \cdot \arrow[lll, "s_1"]
\end{tikzcd}
$$