Poset homology and operads

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joint work with Clément Dupont (IMAG)

Journées du GT LHC
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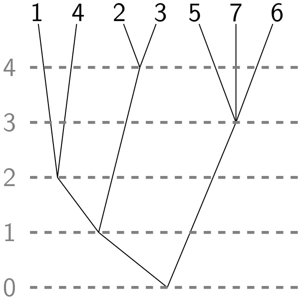
Goal for today

Understand the following relation:

$$H^{n-3}(\Pi_n) = \Sigma\,\mathrm{Lie} \quad \left(H^{n-3}(\mathrm{HT}_n) = \Sigma\,\mathrm{Lie}\right)$$

[Stanley, Hanlon, Joyal(1980s), Fresse (2003), Vallette (2007), ...]



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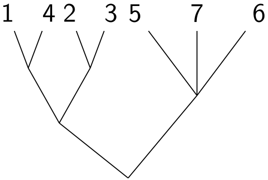


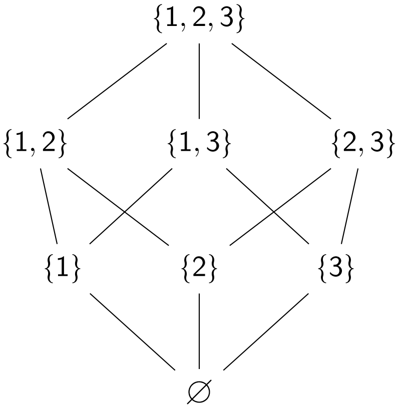
Outline

Posets and chain complexes

First poset (partially ordered set): Boolean poset (or lattice)

Consider the set of subsets of a set V, with the partial order given by inclusion of subsets:

$$A \leqslant B \Leftrightarrow A \subseteq B$$



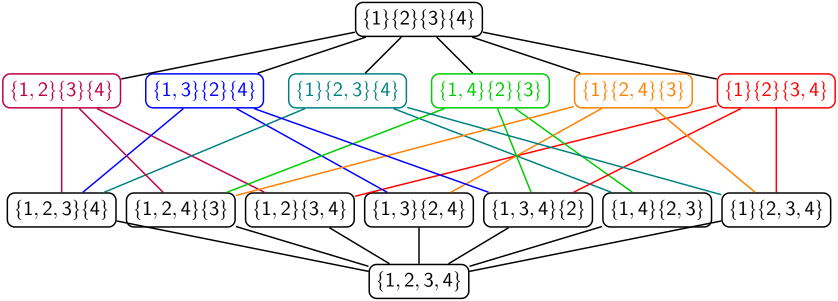
Posets of (set) partitions Π_V

Partitions of a set V :

$$\{V_1, \ldots, V_k\} \models V \Leftrightarrow V = \bigsqcup_{i=1}^{k} V_i, V_i \cap V_j = \varnothing \text{ for } i \neq j$$

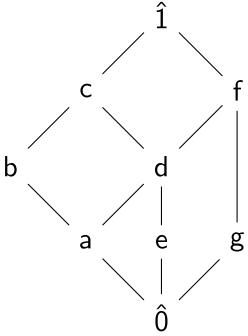
Partial order on set partitions of a set V:

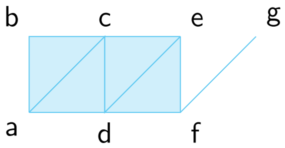
$$\{V_1, \ldots, V_k\} \leqslant \{V'_1, \ldots, V'_p\} \Leftrightarrow \forall i \in \{1, p\}, \exists j \in \{1, k\} \text{ s.t. } V'_i \subseteq V_j$$



Poset cohomology

To any bounded poset P can be associated its order complex (nerve), a simplicial set whose simplices are the k-chains $a_0 < \ldots < a_k$ in $P \backslash \{\hat{0}, \hat{1}\}$. The (co)homology of P is the cohomology of its order complex.





(Co)homology of a poset

Let P be a poset.

$C_j(P) = \mathbb{C}$-vector space of j-chains $x_0 < x_1 < \ldots < x_j$ of $P - \{\hat{0}_P, \hat{1}_P\}$,
with $C_{-1}(P) = \mathbb{C}.e$

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For $j \geqslant 0$, let us define the differential $\partial_j : C_j(P) \to C_{j+1}(P)$ by:

$$\partial(x_0 < \ldots < x_j) = \sum_{i=1}^{j+1} \sum_{x_{i-1} < x < x_i} (-1)^i (x_0 < \ldots < x_{i-1} < x < x_i < \ldots < x_j).$$

We have $\partial_j \partial_{j-1} = 0$: (C_j, ∂_j) is a chain complex.

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We have $\partial_j \partial_{j-1} = 0$: (C_j, ∂_j) is a chain complex.
The jth cohomology group is then defined, for any $j \geqslant 0$, by:

$$\tilde{H}^j(P) = \ker \partial_j / \operatorname{im} \partial_{j-1}.$$

What about unbounded posets ?

→Add bounds to delete them !

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Another definition for the cohomology

$$c^k = \mathbb{C}.\{a_0 < \ldots < a_k | a_0 \text{ minimal and } a_k \text{ maximal}\}$$

In particular, if P is bounded,

$$h^n(P) \simeq \widetilde{H}^{n-2}(P \backslash \{\hat{0}, \hat{1}\}).$$

Cohen-Macaulay posets

Theorem (Björner, 1980)

The partition poset Π_n is Cohen-Macaulay (even EL-shellable): all its cohomology groups vanish but its top one.

→ In this case, the Möbius number gives, up to a sign, the dimension of the unique non trivial cohomology group.

Hence

$$\dim\left(\tilde{\mathrm{H}}^{n-3}(\Pi_n)\right) = (n-1)!$$

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Hence

$$\dim\left(\tilde{\mathrm{H}}^{n-3}(\Pi_n)\right) = (n-1)!$$

Spoiler :

That's the dimension of the vector space $\mathrm{Lie}(n)$!

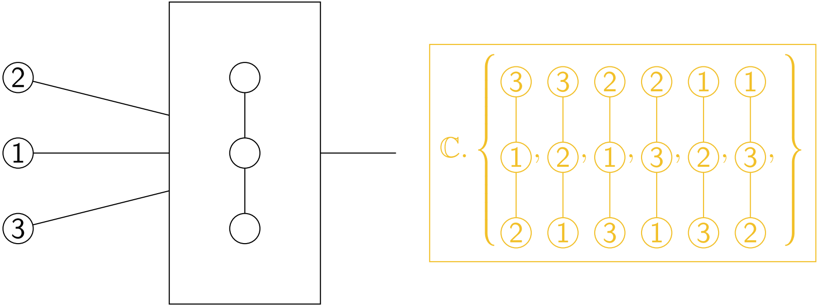
Species and operads

What are species?

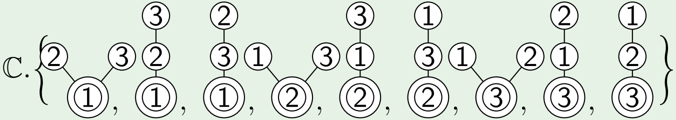
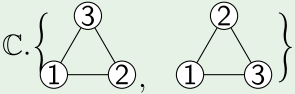
Definition (Joyal, 80s)

A species F is a functor from Bij to Vect. To a finite set S, the species F associates a vector space $\mathrm{F}(S)$ independent from the nature of S.

Species = Construction plan, such that the vector space obtained is invariant by relabeling

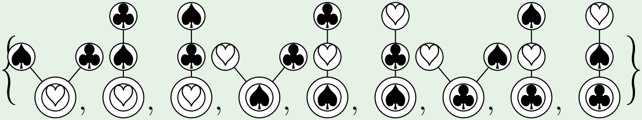
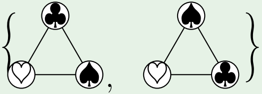


Examples of species

- $\mathbb{C}.\{(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\}$ (Species of lists $\mathbb{L}$ on $\{1,2,3\}$)
- $\mathbb{C}.\{\{1,2,3\}\}$ (Species of non-empty sets $\mathbb{E}^+$)
- $\mathbb{C}.\{\{1\},\{2\},\{3\}\}$ (Species of pointed sets $\mathbb{E}^\bullet$)
-  (Species of Cayley trees $\mathbb{T}$)
-  (Species of cycles)

These sets are the image by species of the set $\{1,2,3\}$.

Examples of species

- $\mathbb{C}.\{(\heartsuit,\spadesuit,\clubsuit),(\heartsuit,\clubsuit,\spadesuit),(\spadesuit,\heartsuit,\clubsuit),(\spadesuit,\clubsuit,\heartsuit),(\clubsuit,\heartsuit,\spadesuit),(\clubsuit,\spadesuit,\heartsuit)\}$ (Species of lists $\mathbb{L}$ on $\{\clubsuit,\heartsuit,\spadesuit\}$)
- $\mathbb{C}.\{\{\heartsuit,\spadesuit,\clubsuit\}\}$ (Species of non-empty sets $\mathbb{E}^{+}$)
- $\mathbb{C}.\{\{\heartsuit\},\{\spadesuit\},\{\clubsuit\}\}$ (Species of pointed sets $\mathbb{E}^{\bullet}$)
- $\mathbb{C}.$  (Species of Cayley trees $\mathbb{T}$)
- $\mathbb{C}.$  (Species of cycles)

These sets are the image by species of the set $\{\clubsuit,\heartsuit,\spadesuit\}$.

Substitution of species

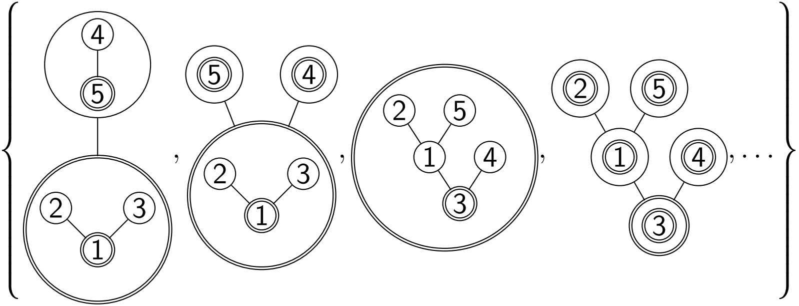
Proposition

Let F and G be two species. Let us define:

$$(F \circ G)(S) = \bigoplus_{\pi \in \Pi(S)} F(\pi) \otimes \bigotimes_{J \in \pi} G(J),$$

where $\Pi(S)$ runs on the set of partitions of S.

$\mathbb{T} \circ \mathbb{T}(\llbracket 5 \rrbracket) =$

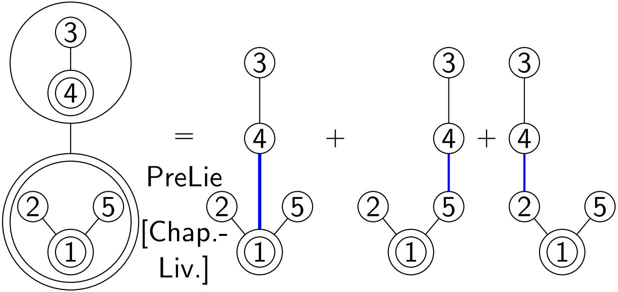
$\mathbb{C}.$ 

Operads

A (symmetric) operad $\mathcal{O}$ is

- a species $\mathcal{O}$ with an associative composition

$$\gamma : \mathcal{O} \circ \mathcal{O} \to \mathcal{O}$$



- and a unit $i : I \to \mathcal{O}$, where I is the singleton species $(I(S) = \delta_{|S|=1}\mathbb{C})$.
- To each kind of algebra is associated an operad.

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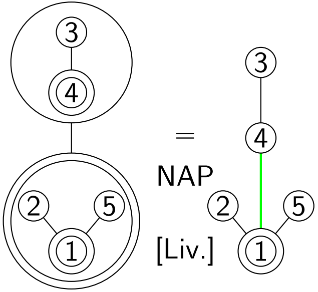

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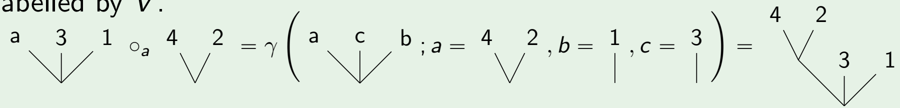
Free operad

Let M be $\mathfrak{S}$-module. The free operad over M is the operad whose underlying species associate to any finite set V the set of rooted trees whose leaves are labelled by V and whose inner vertices are labelled by an element of M, with substitution given by grafting on leaves.

Mag operad

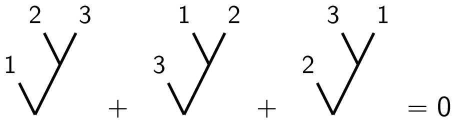
When $M = \mathbb{C}.\{(1,2),(2,1)\}$, the free operad is called Magmatic operad. The species $\mathcal{M}ag(V)$ is the species of planar binary trees with leaves labelled by V.



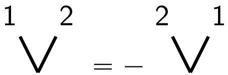
Any operad can be described as a quotient of a free operad.

Lie operad

Lie operad encodes Lie algebra. Its underlying vector space is obtained as a quotient of the Magmatic operad's vector spaces with the Jacobi relations



and the anti-symmetry

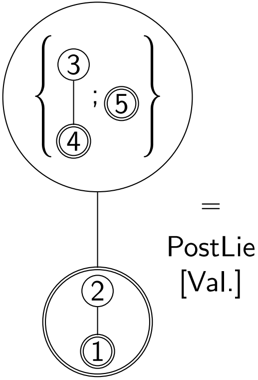


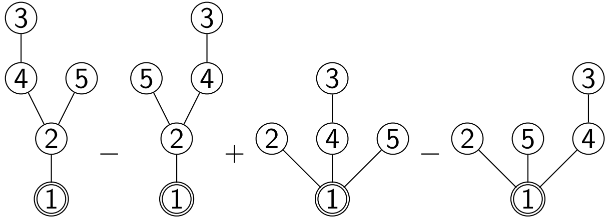
Proposition

The vector space of n-ary operations of Lie operad has dimension $\mathrm{Lie}(n) = (n-1)!$ *(comb).*

Post-Lie operad [Vallette, 07 ; Munthe-Kaas–Wright, 08]

The underlying vector space PostLie(V) of post-Lie operad is spanned by Lie brackets of planar trees with nodes labeled by V. The substitution of a tree t inside a node v is given by the sum over all the way to graft each child of v to the right of a node of t (planar pre-Lie product).





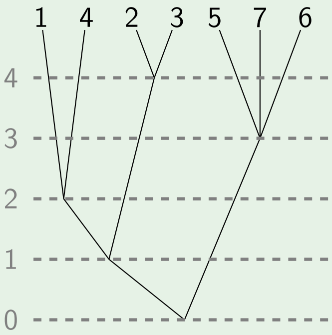
Proposition

The vector space of n-ary operations of Post-Lie operad has dimension $\text{Post-Lie}(n) = \frac{(2n-1)!}{n!}$.

Back to the partition posets and Lie operad

$$C_j(\Pi_n) = \mathbb{C}.\{\hat{0}_{\Pi_n} = \pi_{-1} < \ldots < \pi_{j+1} = \hat{1}_{\Pi_n} | \pi_l \in \Pi_n, \forall l \in [\![-1; j+1]\!]\}$$

Example: leveled cobar construction



Theorem (Fresse, 04)

The action of the symmetric group on the cohomology of the partition posets Π_n is given by

$$\tilde{H}_{n-1}(\Pi_n) = \mathrm{Lie}(n)^\vee \otimes sgn_n$$

where $\mathrm{Lie}(n)^\vee$ is the dual module of Lie.

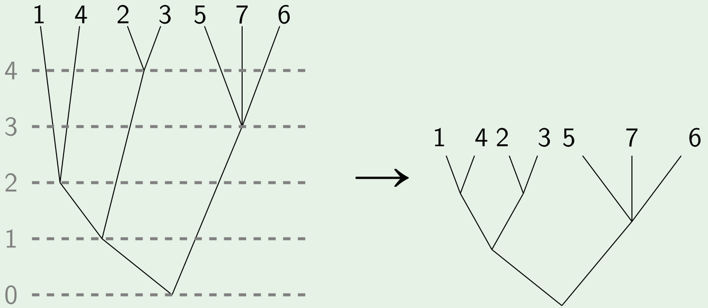
To nested sets

Problem

There are no operadic structure on the leveled cobar construction, but there is one on the cobar construction !

Solution :

Forget about the levels !



This is what we obtain when we consider nested sets instead of chains !

Building sets and nested sets [De Concini–Procesi, 95 ; Feichtner–Müller, 05]

Consider $\mathcal{L}$ a finite join-semilattice (any nonempty subset has a least upper bound). For any $S \subseteq \mathcal{L}$ and $x \in \mathcal{L}$, we write

$$S_{\geqslant X} = \{y \in S | y \geqslant x\}.$$

Definition

A building set is a subset $\mathcal{G}$ in $\mathcal{L}_{<\hat{1}}$ such that for any $x \in \mathcal{L}_{<\hat{1}}$ and $\max \mathcal{G}_{\geqslant x} = \{g_1, \ldots, g_k\}$,there is an isomorphism of posets

$$[x, \hat{1}] \simeq \prod_{i=1}^{k} [g_i, \hat{1}].$$

A nested set is a subset S of $\mathcal{G}$ such that for any set of incomparable elements $x_1, \ldots, x_t$ in S ($t \geqslant 2$), the set $\{x_1, \ldots, x_t\}$ has a greatest lower bound (meet) which does not belong to $\mathcal{G}$.

Topological result

The $\mathcal{G}$-nested sets form an abstract simplicial complex, called the nested set complex.

Proposition (Feichtner–Müller, 05)

Consider a join-semilattice $\mathcal{L}$ and an associated building set $\mathcal{G}$. The associated nested set complex is homotopy equivalent to the order complex of the poset.

For partition posets

The cobar resolution (for the Commutative operad) corresponds to the cochain complex of the nested set complex associated with the minimal building set.

Hypertree posets and postLie operad

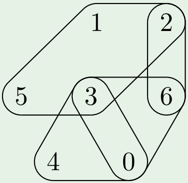
Hypergraphs

Definition (Berge)

A hypergraph (on a set V) is an ordered pair (V, E) where:

- V is a finite set (vertices)
- E is a collection of subsets of cardinality at least two of elements of V (edges).

Example of a hypergraph on $[1; 7]$



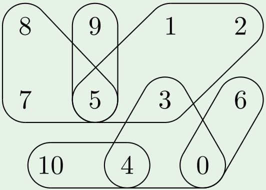
Hypertrees

Definition

A hypertree is a non-empty hypergraph H such that, given any distinct vertices v and w in H,

- there exists a walk from v to w in H with distinct edges e_i, (H is connected),
- and this walk is unique, (H has no cycles).

Example of a hypertree



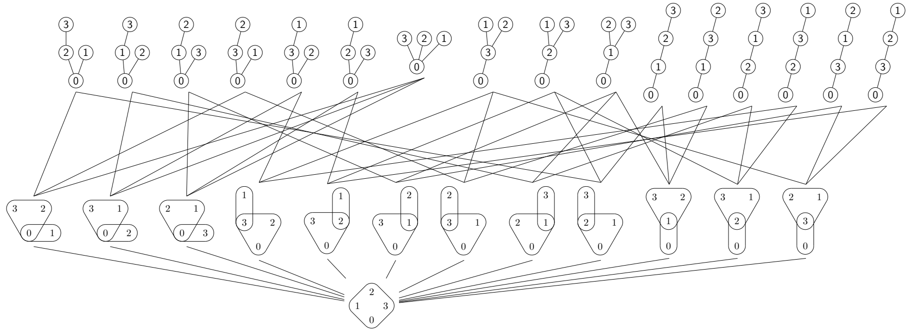
The hypertree poset

Definition

Let I be a finite set of cardinality n, S and T be two hypertrees on I.

$$S \preceq T \Longleftrightarrow \text{Each edge of } S \text{ is the union of edges of } T$$

We write $S \prec T$ if $S \preceq T$ but $S \neq T$.



Euler characteristic of the hypertree posets

Proposition (McCammond-Meier, 2004)

The dimension of the top cohomology group of $\widehat{\mathrm{HT}}_n$ *is given by:*

$$\dim\left(H^{n-2}(\widehat{\mathrm{HT}}_n)\right) = (-1)^{n-1}(n-1)^{n-2}$$

Proposition

The dimension of the top cohomology group of HT_n *is given by:*

$$\dim\left(H^{n-2}(\mathrm{HT}_n)\right) = (-1)^n \frac{(2n-3)!}{(n-1)!}$$

$\frac{(2n-3)!}{(n-1)!}$?

A006963	Number of planar embedded labeled trees with n nodes: (2n-3)!/(n-1)! for n >= 2, a(1) = 1. (Formerly M3076)	28

1, 1, 3, 20, 210, 3024, 55440, 1235520, 32432400, 980179200, 33522128640, 1279935820800, 53970627110400, 2490952020480000, 124903451312640000, 6761440164390912000, 393008709555221760000, 24412776311194951680000, 1613955767240110694400000 (list; graph; refs; listen; history; text; internal format)

OFFSET 1,3

COMMENTS For n>1: central terms of the triangle in A173333; cf. A001761, A001813. - Reinhard Zumkeller, Feb 19 2010
Can be obtained from the Vandermonde permanent of the first n positive integers; see A093883. - Clark Kimberling, Jan 02 2012
All trees can be embedded in the plane, but "planar embedded" means that orientation matters but rotation doesn't. For example, the n-star with n-1 edges has n! ways to label it, but rotation removes a factor of n-1. Another example, the n-path has n! ways to label it, but rotation removes a factor of 2. - Michael Somos, Aug 19 2014

REFERENCES N. J. A. Sloane and Simon Plouffe, The Encyclopedia of Integer Sequences, Academic Press, 1995 (includes this sequence).

LINKS Vincenzo Librandi, Table of n, a(n) for n = 1..200
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Bradley Robert Jones, On tree hook length formulas, Feynman rules and B-series, Master's thesis, Simon Fraser University, 2014.
Pierre Leroux and Brahim Miloudi, Généralisations de la formule d'Otter, Ann. Sci.

Maximal intervals in HT_n are join-semilattices

Lemma

The cartesian product of join-semilattices is a join-semilattice.

Lemma

$$\mathrm{HT}_n^a = \prod_{v \in V(a)} \Pi_{\mathrm{deg}(v)}$$

Proposition

Every maximal interval HT_n^a in the hypertree posets is a join-semilattice.

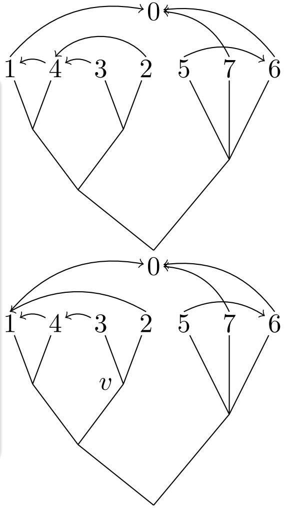
The nested set complex of hypertrees

The nested sets of hypertrees are the following combinatorial objects:

Definition

A merge tree is a pair (T, τ) of trees such that

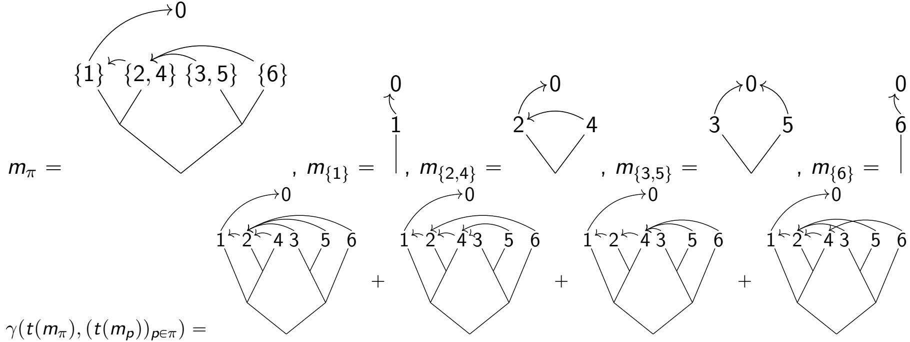
- T is a (non planar) rooted reduced (no vertex of valency 2) tree with leaves labeled by $\{1, \ldots, n\}$
- τ is a (non planar oriented) tree whose vertices are labeled by $\{0, \ldots, n\}$ and whose root is 0
- for any internal vertex s in T, the restriction of τ to edges leaving the leaves above s is connected



Operadic composition

The operadic composition of a bitree b in a node v is as follows:

- the blue children of v are grafted to some nodes in b (pre-Lie composition)
- the bottom tree of b is grafted at the place of the leaf v (usual magmatic composition)

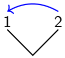


Operadic structure on the cohomology of the nested set complex (aka. post-Lie !)

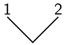
Let us consider the map

$$\text{Post-Lie} \xrightarrow{\phi} H^\bullet(HT_\bullet)$$

$$1 \lhd 2 \mapsto$$



$$\{1;2\} \mapsto$$



Theorem (DO–Dupont, 22+)

The map ϕ is an operad morphism. The cohomology of the hypertree poset can be endowed with an operadic structure. It is then isomorphic to the suspension of post-Lie operad.

Where does the idea of the composition come from ?

Let us consider the map $a : \mathrm{HT}_n^0 \to \Pi_n$. We define

$$(\mathrm{HT})_{\leqslant\pi} := a^{-1}\left(\Pi_{\leqslant\pi}\right) \text{ and } (\mathrm{HT})_{\geqslant\pi} := a^{-1}\left(\Pi_{\geqslant\pi}\right).$$

Define the maps

$$\varphi : (\mathrm{HT})_{\leqslant\pi} \to \mathrm{HT}(\pi)$$

and

$$\psi : (\mathrm{HT})_{\geqslant\pi} \to \prod_{t\in\pi} \mathrm{HT}(t)$$

obtained respectively by contracting parts of π to an element and splitting the hypertree according to the parts of π.

The idea is to use these maps to define a composition:

$$C^\bullet\left(\mathrm{HT}(\pi)\right) \otimes \bigotimes_{T\in\pi} C^\bullet\left(\mathrm{HT}(T)\right) \simeq C^\bullet\left(\mathrm{HT}(\pi)\right) \otimes C^\bullet\left(\prod_{T\in\pi} \mathrm{HT}(T)\right)$$

$$\xrightarrow{\phi^*\otimes\psi^*} C^\bullet\left(\mathrm{HT}_{\leqslant\pi}\right) \otimes C^\bullet\left(\mathrm{HT}_{\geqslant\pi}\right) \to C^\bullet\left(\mathrm{HT}_n\right)$$

Finally

Other results proven and to come

- We obtained an operad on the nested sets which is a model of (the suspension of) postLie.
- By considering chains from the minimal element to anywhere, we prove that preLie operad as a left post-lie module structure.

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$$(G \leftharpoondown D) \lhd T = (G \lhd T) \leftharpoondown D + G \leftharpoondown (D \lhd T)$$
$$\{S, T\} = T \leftharpoondown S - S \leftharpoondown T,$$

  where $\leftharpoondown$ is the usual pre-Lie product.
- The construction of last slide can be applied to many other examples : bidecorated partition posets, bidecorated hypertree posets, ...

Finally

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Thank you for your attention !