The homotopy theory of operated algebras

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Plan

- Motivation: Deformation theory and homotopy theory
- Differential graded Lie algebras (\(=\)dg Lie algebras) vs \(L_{\infty}\)-algebras
- Homotopy theory of differential (associative) algebras
  - Formal deformations and cohomology theory of differential algebras
  - \(L_{\infty}\)-structures
  - homotopy Rota-Baxter algebras
  - minimal model
- Other operated algebras
Part I: Philosophy of Deformation Theory after Deligne, Drinfeld, Kontsevich,…

▶ (Deligne 1986): "The deformation theory of any mathematical object, e.g., an associative algebra, a complex manifold, etc., can be described starting from a certain dg Lie algebra associated to the mathematical object in question."
Part I: Philosophy of Deformation Theory after Deligne, Drinfeld, Kontsevich,…

► (Deligne 1986): "The deformation theory of any mathematical object, e.g., an associative algebra, a complex manifold, etc., can be described starting from a certain dg Lie algebra associated to the mathematical object in question."

Theorem (Lurie, Pridham)

In characteristic 0, there exists an equivalence between the $\infty$-category of formal moduli problems and the $\infty$-category of DG Lie algebras ($L_\infty$-algebras).

J. Lurie, DAG X: Formal moduli problems.

Part I: Problems from Deformation Theory

Given an algebraic structure governed by an operad $\mathcal{P}$, two basic problems of deformation theory:

**Problem**
*Find a homotopy version (or minimal model) $\mathcal{P}_\infty$*

**Problem**
*Define the deformation cohomology of $\mathcal{P}$-algebras and describe the $L_\infty$ structure on the deformation complex*
Part I: Koszul case

Assume the operad $\mathcal{P}$ is Koszul.

Example

- associative algebras
- commutative associative algebras
- Lie algebras
- Poisson algebras
- pre-Lie algebras
- Leibniz algebras
- Lie triple systems
- etc
Part I: Koszul case

Problem

Find a homotopy version (or minimal model) $\mathcal{P}_\infty$

Answer

$\mathcal{P}_\infty = \Omega(\mathcal{P}^i)$ is the minimal model of $\mathcal{P}$.


Part I: From minimal models to $L_\infty$-structures

Problem
Describe the $L_\infty$ structure on the deformation complex

Answer
Given a cofibrant resolution, or in particular, the minimal model $P_\infty$ of $P$, one can define the deformation cohomology and describe the $L_\infty$ structure on the deformation complex.

Part I: Koszul case

Given a Koszul operad $\mathcal{P}$,

minimal model $\mathcal{P}_\infty = \Omega(\mathcal{P}^i)$

$\Downarrow$

deformation complex $\text{Hom}(\mathcal{P}^i, \text{End}_V)$

and

dg Lie algebra structure on the deformation complex
Part I: Non-Koszul case

When \( \mathcal{P} \) is NOT Koszul, no general answer so far.

Example

- Rota-Baxter associative/Lie algebras
- differential associative/Lie algebras with nonzero weight
- Hom-associative algebras, Hom-Lie algebras, ···
- etc
Part I: Differential algebras

Definition
Let \( \lambda \in k \) be a fixed element. A differential algebra of weight \( \lambda \) is an associative algebra \((A, \mu_A)\) together with a linear operator \( d_A : A \to A \) such that

\[
d_A(ab) = d_A(a)b + ad_A(b) + \lambda \ d_A(a)d_A(b), \quad \forall a, b \in A.
\]

Example
Let \( A = C^\infty(\mathbb{R}) \) be the algebra of smooth functions on \( \mathbb{R} \). Let \( d \) be the classical derivation operation of smooth functions. Then \( A \) is a differential algebra of weight 0.
Remark
The defining relation of differential operators of any weight is given by

\[ d_A \circ \mu_A = \mu_A \circ (d_A \otimes \text{Id} + \text{Id} \otimes \mu_A) + \lambda \mu_A \circ (d_A \otimes d_A) \]

defined in terms of maps. If \( \lambda = 0 \), the operad of differential algebras of weight zero is Koszul, as shown by Loday in 2010. When \( \lambda \neq 0 \), this relation is NOT quadratic and the operad of differential algebras of nonzero weight is not quadratic and not even homogeneous, so the Koszul duality theory for operads could not be applied directly to develop a cohomology theory of differential algebras of any weight.

Question (Loday)
If the parameter \( \lambda \) is different from 0, then the operad \( \lambda \text{-AsDer} \) is not a quadratic operad since the term \( d(a)d(b) \) needs three generating operations to be defined. So one needs new techniques to extend Koszul duality to this case.

Part I: Non-Koszul case

Four steps:

1. Formal deformations
2. Deformation complex
3. $L_\infty$-structure
4. Minimal model
Part II: Differential graded Lie algebras

Throughout this talk, let $k$ be a field of characteristic zero.

**Definition**

A differential graded Lie algebra (aka dg Lie algebra) is a graded space $L = \bigoplus_{i \in \mathbb{Z}} L_i$ together with two operations:

- $l_1 : L_i \to L_{i-1}$
- $l_2 : L_i \otimes L_j \to L_{i+j}$

of degree $-1$ and $0$ such that

(i) $l_1 : L_i \to L_{i-1}$ is a differential,

(ii) $l_2 : L_i \otimes L_j \to L_{i+j}$ is a Lie bracket,

(iii) $l_1$ is a derivation for $l_2$, i.e.

$$l_1 l_2(a \otimes b) = l_2(l_1(a) \otimes b) + (-1)^{|a|} l_2(a \otimes l_1(b))$$

for $a, b \in L$ homogeneous.
Part II: Maurer-Cartan elements in dg Lie algebras

Definition
Let \( L \) be a dg Lie algebra. An element \( \alpha \in L_{-1} \) is a Maurer-Cartan element if
\[
l_1(\alpha) - \frac{1}{2} l_2(\alpha \otimes \alpha) = 0.
\]

Proposition (Twisting procedure)
Let \( L \) be a dg Lie algebra. Given a Maurer-Cartan element \( \alpha \in L_{-1} \), one can produce a new dg Lie algebra by imposing
\[
l_1^\alpha(x) = l_1(x) - l_2(\alpha \otimes x)
\]
and
\[
l_2^\alpha(x \otimes y) = l_2(x \otimes y)
\]
Part II: $L_\infty$-algebras

Definition
Let $L = \bigoplus L_i$ be a graded space over $k$. Assume that $L$ is endowed with a family of linear operators $l_n : L^\otimes n \to L$, $n \geq 1$ with $|l_n| = n - 2$ satisfying the following conditions: $\forall \sigma \in S_n, x_1, \ldots, x_n \in L$,

(i) (Skew-symmetry)

$$l_n(x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(n)}) = \chi(\sigma, x_1, \ldots, x_n) l_n(x_1, \ldots, x_n),$$

(ii) (Higher Jacobi identities)

$$\sum_{i=1}^{n} \sum_{\sigma \in Sh(i, n-i)} \chi(\sigma, x_1, \ldots, x_n)(-1)^{(n-i)} l_{n-i+1}(l_i(x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(i)}) \otimes x_{\sigma(i+1)} \otimes \ldots \otimes x_{\sigma(n)}) = 0,$$

where $Sh(i, n-i)$ is the set of $(i, n-i)$ shuffles, i.e.,

$$Sh(i, n-i) = \{\sigma \in S_n \text{ such that } \sigma(1) < \sigma(2) < \ldots < \sigma(i), \text{ and } \sigma(i+1) < \sigma(i+2) < \ldots \sigma(n)\}.$$

Then $(L, \{l_n\}_{n \geq 1})$ is called a $L_\infty$-algebra.
Part II: Maurer-Cartan elements and $L_\infty$-algebras

Definition

Let $(L, \{l_n\}_{n \geq 1})$ be an $L_\infty$-algebra and $\alpha \in L_{-1}$. Then $\alpha$ is called a Maurer-Cartan element if it satisfies equation:

$$\sum_{n=1}^{\infty} \frac{1}{n!} (-1)^{\frac{n(n-1)}{2}} l_n(\alpha^{\otimes n}) = 0,$$

whenever this infinite sum exists.

Proposition

Let $(L, \{l_n\}_{n \geq 1})$ be an $L_\infty$-algebra. Given a Maurer-Cartan element $\alpha$ in $L_\infty$-algebra $L$, we can define a new $L_\infty$ structure $\{l_n^\alpha\}_{n \geq 1}$ on graded space $L$, where $l_n^\alpha : L^{\otimes n} \to L$ is defined as:

$$l_n^\alpha(x_1 \otimes \ldots \otimes x_n) = \sum_{i=0}^{\infty} \frac{1}{i!} (-1)^{i n + \frac{i(i-1)}{2}} l_{n+i}(\alpha^{\otimes i} \otimes x_1 \otimes \ldots \otimes x_n).$$
Part III: Differential algebras

Definition
Let $\lambda \in \mathbf{k}$ be a fixed element. A differential algebra of weight $\lambda$ is an associative algebra $(A, \mu_A)$ together with a linear operator $d_A : A \to A$ such that

$$d_A(ab) = d_A(a)b + ad_A(b) + \lambda d_A(a)d_A(b), \quad \forall a, b \in A.$$

Remark
The defining relation of differential operators of any weight is given by

$$d_A \circ \mu_A = \mu_A \circ (d_A \otimes \text{Id} + \text{Id} \otimes d_A) + \lambda \mu_A \circ (d_A \otimes d_A)$$

expressed in terms of maps. If $\lambda = 0$, the operad of differential algebras of weight zero is Koszul, but while $\lambda \neq 0$, it is NOT Koszul.
Definition

Let \((A, \mu_A, d_A)\) be a differential algebra. A 1-parameterized family

\[ d_t = \sum_{i=0}^{\infty} d_i t^i, \quad d_i \in \text{Hom}(A, A). \]

is called a \textbf{1-parameter formal deformation} of the differential operators \(d_A\) if \(d_t\) is a differential operators of weight \(\lambda\) on the associative algebra \((A[[t]], \mu_A)\) such that \(d_0 = d_A\).

For all \(x, y, z \in A\), the following equalities hold:

\[ d_t(\mu_t(x, y)) = \mu_A(d_t(x), y) + \mu_A(x, d_t(y)) + \lambda \mu_A(d_t(x), d_t(y)). \]
For any $n \geq 1$, we have

$$d_n \mu_A(x, y) = \mu_A(d_n(x), y) + \mu_A(x, d_n(y)) + \lambda \sum_{l, m \geq 0 \atop l + m = n} \mu_A(d_l(x), d_m(y)).$$

Consider the case $n = 1$,

$$(x + \lambda d_A(x))d_1(y) - d_1(xy) + d_1(x)(y + \lambda d_A(y)) = 0. \quad (1)$$
Part III: New structures arising from a differential algebra

Let \((A, \mu_A, d_A)\) be a differential algebra of weight \(\lambda\). Let \(M\) be a differential bimodule.

**Proposition**

We define a left action “\(\triangleright\)" and a right action “\(\triangleleft\)" of \(A\) on \(M\) as follows: for any \(a \in A, x \in M,\)

\[
a \triangleright x : = ax + \lambda d_A(a)x, \\
x \triangleleft a : = xa + \lambda xd_A(a).
\]

(2) (3)

Then “\(\triangleright\)" and “\(\triangleleft\)" make \(M\) into a bimodule over \(A\) and denote this new bimodule by \(\triangleright M \triangleleft\).

Deformation equation:

\[
x \triangleright d_1(y) - d_1(xy) + d_1(x) \triangleleft y = 0 \iff \partial_{\text{Alg}}(d_1) = 0 \in C^2_{\text{Alg}}(A, \triangleright A \triangleleft).\]

(4)
Definition
Let \((A, \mu_A, d_A)\) be a differential algebra of weight \(\lambda\). Let \(M\) be a differential bimodule. Then the cochain complex

\[ C_{\text{DO}_\lambda}(A, M) =: (C_{\text{Alg}}(A, \triangleright M\triangleleft), \partial), \]

i.e, the Hochschild cochain complex of \(A\) with coefficients in \(\triangleright M\triangleleft\), is called the cochain complex of differential operator \(d\) with coefficients in the differential bimodule \(M\).

The cohomology groups of \(C_{\text{DO}_\lambda}(A, \mu, d)\) are called the cohomology groups of differential operator \(d\), denoted by \(H_{\text{DO}_\lambda}(A)\).
Definition

Let $(A, \mu_A, d_A)$ be a differential algebra. A 1-parameterized family

$$\mu_t = \sum_{i=0}^{\infty} \mu_i t^i, \quad \mu_i \in \text{Hom}(A \otimes A, A), \quad d_t = \sum_{i=0}^{\infty} d_i t^i, \quad d_i \in \text{Hom}(A, A).$$

is called a 1-parameter formal deformation of the differential algebra $(A, d_A)$ if the pair $(\mu_t, d_t)$ endows the $k[[t]]$-module $(A[[t]], \mu_t, d_t)$ with a differential algebra structure over $k[[t]]$ such that $(\mu_0, d_0) = (\mu_A, d_A)$.

For all $x, y, z \in A$, the following equalities hold:

$$\mu_t(\mu_t(x, y), z) = \mu_t(x, \mu_t(y, z)),$$

$$d_t(\mu_t(x, y)) = \mu_t(d_t(x), y) + \mu_t(x, d_t(y)) + \lambda \mu_t(d_t(x), d_t(y)).$$
Part III: Formal deformations of differential algebras

For any $n \geq 1$, we have

$$\sum_{i}^{n} \mu_{i}(\mu_{n-i}(x, y), z) = \sum_{i}^{n} \mu_{i}(x, \mu_{n-i}(y, z)),$$

$$\sum_{\substack{k, l \geq 0 \\ k+l=n}} d_{l} \mu_{k}(x, y) = \sum_{\substack{k, l \geq 0 \\ k+l=n}} (\mu_{k}(d_{l}(x), y) + \mu_{k}(x, d_{l}(y))) + \lambda \sum_{\substack{k, l, m \geq 0 \\ k+l+m=n}} \mu_{k}(d_{l}(x), d_{m}(y)).$$

Consider the case $n = 1$,

$$x \mu_{1}(y, z) - \mu_{1}(xy, z) + \mu_{1}(x, yz) - \mu_{1}(x, y)z = 0,$$

$$x \uplus d_{1}(y) - d_{1}(xy) + d_{1}(x) \triangleleft y = d_{A} \mu_{1}(x, y) - \lambda \mu_{1}(d_{A}(x), d_{A}(y)).$$
Part III: A chain map

Define a chain map \( \Phi^* : C_{\text{Alg}}^*(A, M) \to C_{\text{DO}_\lambda}^*(A, M) \) as follows: for any \( f \in C_{\text{Alg}}^n(A, M) \) with \( n \geq 1 \),

\[
\Phi^n(f)(a_{1,n}) = \sum_{k=1}^{n} \lambda^{k-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} f(a_{1,i_1-1}, d_A(a_{i_1}), a_{i_1+1,i_2-1}, d_A(a_{i_2}), \cdots, d_A(a_{i_k}), a_{i_k+1,n}) - d_M(f(a_{1,n}))
\]

and

\[
\Phi^0(x) = -d_M(x), \quad \forall \ x \in C_{\text{Alg}}^0(A, M) = M.
\]
Part III: Return to formal deformations

Consider the case $n = 1$,

$$x\mu_1(y, z) - \mu_1(xy, z) + \mu_1(x, yz) - \mu_1(x, y)z = 0 \iff \partial^2_{\text{Alg}} \mu_1 = 0,$$

$$x \triangleright d_1(y) - d_1(xy) + d_1(x) \triangleleft y = d_A \mu_1(x, y) - \lambda \mu_1(d_A(x), d_A(y)) \iff \partial^2_{\text{DO}_\lambda} d_1 + \Phi^2 \mu_1 = 0$$

Proposition

Case $n = 1 \iff \partial^2_{\text{DA}_\lambda} (\mu_1, d_1) = 0$. 
Definition

The negative shift of the mapping cone of $\Phi^*$, denoted by

$$(C^*_{DA,\lambda}(A, M), \partial^*_{DA,\lambda})$$

is called the cochain complex of the differential algebra $A$ with coefficients in the differential bimodule $M$.

When the differential bimodule $M$ is the regular differential bimodule $A$, we write $$(C^*_{DA,\lambda}(A), \partial^*_{DA,\lambda}):= (C^*_{DA,\lambda}(A, M), \partial^*_{DA,\lambda}).$$
Part III: $L_\infty$-algebra structure on $\mathcal{C}_{DA\lambda}(V)$

Let $V$ be a graded space. Define a graded space $\mathcal{C}_{DA\lambda}(V)$ as :

$$\mathcal{C}_{DA\lambda}(V) = \mathcal{C}_{\text{Alg}}(V) \oplus \mathcal{C}_{\text{DO}\lambda}(V),$$

where

$$\mathcal{C}_{\text{Alg}}(V) = \text{Hom}(T^c(sV), sV), \mathcal{C}_{\text{DO}\lambda}(V) = \text{Hom}(T^c(sV), V).$$

**Theorem**

Given a graded space $V$ and an element $\lambda \in k$, there is an $L_\infty$-algebra structure on graded space

$$\mathcal{C}_{DA\lambda}(V)$$

such that it becomes Gerstenhaber graded Lie algebra over $\mathcal{C}_{\text{Alg}}(V)$. 
Part III: $L_\infty$-algebra structure on $\mathcal{C}_{DA,\lambda}(V)$

**Theorem**

Let $V$ be an ungraded vector space.

- The set of Maurer-Cartan elements in the $L_\infty$-algebra $\mathcal{C}_{DA,\lambda}(V)$ is in bijection with the differential algebra structure of weight $\lambda$ on $V$.
- Given a Maurer-Cartan element $\alpha$ in $\mathcal{C}_{DA,\lambda}(V)$, the underlying complex of the twisted $L_\infty$-algebra by this Maurer-Cartan element is just the cochain complex of the corresponding differential algebra.
Part III: Homotopy differential algebras

Definition

Let $V$ be a graded space. A homotopy differential algebra structure of weight $\lambda$ on $V$ is defined to be a Maurer-Cartan element in the $L_\infty$-algebra $\mathcal{C}_{DA,\lambda}(V)$. 
Part III: Homotopy differential algebras

Definition

Let $V$ be a graded space. Then a homotopy differential algebra structure of weight $\lambda$ on $V$ consists of two families of operators $\{m_n\}_{n\geq 1}, \{d_n\}_{n\geq 1}$ with $m_n : V \otimes^n \to V, |m_n| = n - 2, d_n : V \otimes^n \to V, |d_n| = n - 1$ satisfying the following two conditions:

(1)

$$\sum_{1 \leq i+j \leq n} (-1)^{i+j+k} m_{n-j+1} \circ \left( \text{id} \otimes^i \otimes m_j \otimes \text{id} \otimes^{n-i-j} \right) = 0,$$

(2)

$$\sum_{i+j+k=n} (-1)^{i+j+k} d_{n-j+1} \left( \text{id} \otimes^i \otimes m_j \otimes \text{id} \otimes^k \right)$$

$$= \sum_{j_1 + \cdots + j_{q+1} + q = p, \ 1 + \cdots + l_q + j_1 + \cdots + j_{q+1} = n} (-1)^{\eta \lambda q - 1} m_p \circ \left( \text{id} \otimes^{j_1} \otimes d_{l_1} \otimes \text{id} \otimes^{j_2} \otimes \cdots \otimes d_{l_q} \otimes \text{id} \otimes^{j_{q+1}} \right),$$
Part III: Homotopy differential algebras

(i) when $n = 1$, $|d_1| = 0$ and

$$d_1 \circ m_1 = m_1 \circ d_1,$$

that is, $d_1 : (V, m_1) \rightarrow (V, m_1)$ is a chain map;

(ii) when $n = 2$, $|d_2| = 1$ and and

$$d_1 \circ m_2 - \left( m_2 \circ (d_1 \otimes \text{Id}) + m_2 \circ (\text{Id} \otimes d_1) + \lambda m_2 \circ (d_1 \otimes d_1) \right)$$

$$= d_2 \circ (\text{Id} \otimes m_1 + m_1 \otimes \text{Id}) + m_1 \circ d_2,$$

which shows that $d_1$ is, up to a homotopy given by $d_2$, a differential operator of weight $\lambda$ with respect to the “multiplication” $m_2$.  

Part III: Minimal model

Theorem
The dg operad $\mathcal{D}if_\infty$ of homotopy differential algebras of weight $\lambda$ is the minimal model of the operad $\mathcal{D}if$ of differential algebras of weight $\lambda$, that is, there is a surjective quasi-isomorphism of operads $p : \mathcal{D}if_\infty \to \mathcal{D}if$ subject to a certain minimality condition.

Remark
One can introduce the Koszul dual $\mathcal{D}if^i$ such that $\Omega(\mathcal{D}if^i) = \mathcal{D}if_\infty$, but $\mathcal{D}if^i$ is only a homotopy cooperad.

Part III: Minimal model

As a free graded operad, the dg operad $\mathcal{Dif}_\infty$ is generated by

\[
\begin{array}{cc}
1 \cdots i \cdots n & 1 \cdots i \cdots n \\
m_n(n \geq 2) & d_n(n \geq 1)
\end{array}
\]
Part III: Minimal model

The differential is given by

\[ \partial m_n(n \geq 2) = \sum \pm m_{n-j+1} \]

\[ \partial d_n(n \geq 1) = \pm d_{n-j+1} + \sum \pm \lambda^{q-1} \]

\[ + \sum \pm \lambda^{q-1} \]

\[ \partial m_p \]
Part IV: (Relative) Rota-Baxter associative/Lie algebras

Definition
Let $(R, \mu = \cdot)$ be an associative algebra and $\lambda \in k$. A linear operator $T : R \to R$ is said to be a **Rota-Baxter operator of weight** $\lambda$ if it satisfies

$$\mu \circ (T \otimes T) = T \circ \mu \circ (Id \otimes T) + T \circ \mu \circ (T \otimes Id) + \lambda T \circ \mu.$$  \hfill (7)

Then $(R, \mu, T)$ is called a **Rota-Baxter algebra of weight** $\lambda$.

Definition
A relative Rota-Baxter Lie algebra is a triple $((g, [-, -]_g), \rho, T)$, where $(g, [-, -]_g)$ is a Lie algebra, $\rho : g \to gl(V)$ is a representation of $g$ on a vector space $V$ and $T : V \to g$ is a relative Rota-Baxter operator, i.e.

$$[Tu, Tv]_g = T(\rho(Tu)(v) - \rho(Tv)(u)), \forall u, v \in V$$
Part IV: (Relative) Rota-Baxter associative/Lie algebras

- Can define the deformation complex and construct the $L_\infty$-structure
- Can define homotopy version


Part IV: (Relative) Rota-Baxter associative/Lie algebras

- Can prove the minimal model


Part IV: Averaging algebras and embedding tensors

Definition
Let $R$ be an associative algebra over field $k$. An averaging operator over $R$ is a $k$-linear map $A : R \to R$ such that

$$A(x)A(y) = A(A(x)y) = A(xA(y))$$

for all $x, y \in R$.

Definition
Let $\mathfrak{g}$ be a Lie algebra over a field $k$ and $V$ a representation of $\mathfrak{g}$. A $k$-linear map $A : V \to \mathfrak{g}$ is an embedding tensor if

$$[A(x), A(y)] = A(A(x)y)$$

for all $x, y \in V$. 
Part IV: Averaging algebras and embedding tensors

- Deformation complex, and $L_\infty$-structure on deformation complex


- Minimal model?
Part IV: Nijenhuis algebras

Definition
Let \((A, \mu = \cdot)\) be an associative algebra over field \(k\). A linear operator \(P : A \to A\) is said to be a Nijenhuis operator if it satisfies

\[
\mu \circ (P \otimes P) = P \circ (\mu \circ (\text{Id} \otimes P) + \mu \circ (P \otimes \text{Id}) - P \circ \mu).
\]

In this case, \((A, \mu, P)\) is called a Nijenhuis algebra.

Theorem
- Can define deformation complex of Nijenhuis algebras
- Can construct \(L_\infty\)-structure on the deformation complex and define homotopy Nijenhuis algebras
- Can prove the minimal model
- Have some applications to Nijenhuis geometry


Thank you very much!