The homotopy theory of operated algebras

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LHC days

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Papers

- L. Guo, Y. Li, Y. Sheng and G. Zhou, *Cohomologies, extensions and deformations of differential algebras with any weights.* Theory Appl. Categ. Vol. 38, 2022, No. 37, pp 1409-1433.
- J. Chen, L. Guo, K. Wang and G. Zhou, *Koszul duality, minimal model and L_∞-structure for differential algebras with weight*. arXiv:2302.13216.
- K. Wang and G. Zhou, *Deformations and homotopy theory of Rota-Baxter algebras of any weight*. arXiv:2108.06744.
- K. Wang and G. Zhou, *The homotopy theory and minimal model of Rota-Baxter algebras of arbitrary weight*. arXiv:2203.02960.

Plan

- Motivation: Deformation theory and homotopy theory
- Differential graded Lie algebras (=dg Lie algebras) vs L_∞-algebras
- Homotopy theory of differential (associative) algebras
  - Formal deformations and cohomology theory of differential algebras
  - L_∞-structures
  - homotopy Rota-Baxter algebras
  - minimal model
- Other operated algebras

Part I: Philosophy of Deformation Theory after Deligne, Drinfeld, Kontsevich,...

- (Deligne 1986): "The deformation theory of any mathematical object, e.g., an associative algebra, a complex manifold, etc., can be discribed starting from a certain dg Lie algebra associated to the mathematical obejct in question."

Part I: Philosophy of Deformation Theory after Deligne, Drinfeld, Kontsevich,...

- (Deligne 1986): `"The deformation theory of any mathematical object, e.g., an associative algebra, a complex manifold, etc., can be discribed starting from a certain dg Lie algebra associated to the mathematical obejct in question."`

Theorem (Lurie, Pridham)

In characteristic 0, there exists an equivalence between the ∞-category of formal moduli problems and the ∞-category of DG Lie algebras (L_∞-algebras).

J. Lurie, DAG X: Formal moduli problems.

J. P. Pridham, *Unifying derived deformation theories*, Adv. Math. **224** (2010), no. 3, 772-826.

Part I: Problems from Deformation Theory

Given an algebraic structure governed by an operad $\mathcal{P}$, two basic problems of deformation theory:

Problem

Find a homotopy version (or minimal model) $\mathcal{P}_\infty$

Problem

Define the deformation cohomology of $\mathcal{P}$*-algebras and describe the* L_∞ *structure on the deformation complex*

Part I: Koszul case

Assume the operad $\mathcal{P}$ is Koszul.

Example

- *associative algebras*
- *commutative associative algebras*
- *Lie algebras*
- *Poisson algebras*
- *pre-Lie algebras*
- *Leibniz algebras*
- *Lie triple systems*
- *etc*

Part I: Koszul case

Problem

Find a homotopy version (or minimal model) $\mathcal{P}_\infty$

Answer

$\mathcal{P}_\infty = \Omega(\mathcal{P}^{\text{i}})$ *is the minimal model of* $\mathcal{P}$*.*

V. Ginzburg, M. Kapranov, *Koszul duality for operads.* Duke Math. J. **76** (1994), no. 1, 203-272.

E. Getzler and D. S. J. Jones, *Operads, homotopy algebra and iterated integrals for double loop spaces*, hep-th/9403055 (1994).

Part I: From minimal models to L_∞-structures

Problem

Describe the L_∞ structure on the deformation complex

Answer

Given a cofibrant resolution, or in particular, the minimal model $\mathcal{P}_\infty$ of $\mathcal{P}$, one can define the deformation cohomology and describe the L_∞ structure on the deformation complex.

M. Kontsevich and Y. Soibelman, *Deformations of algebras over operads and the Deligne conjecture*, Conférence Moshé Flato 1999, Vol. I (Dijon), Math. Phys. Stud. 21 (2000), 255-307.

Part I: Koszul case

Given a Koszul operad $\mathcal{P}$,

$$\text{minimal model } \mathcal{P}_\infty = \Omega(\mathcal{P}^{\text{i}})$$
$$\Downarrow$$
$$\text{deformation complex } \mathrm{Hom}(\mathcal{P}^{\text{i}}, \mathrm{End}_V)$$

and

dg Lie algebra structure on the deformation complex

Part I: Non-Koszul case

When $\mathcal{P}$ is NOT Koszul, no general answer so far.

Example

- *Rota-Baxter associative/Lie algebras*
- *differential associative/Lie algebras with nonzero weight*
- *Hom-associative algebras, Hom-Lie algebras,* $\cdots$
- *etc*

Part I: Differential algebras

Definition

Let $\lambda \in \mathbf{k}$ be a fixed element. A differential algebra of weight λ is an associative algebra (A, μ_A) together with a linear operator $d_A : A \to A$ such that

$$d_A(ab) = d_A(a)b + a d_A(b) + \lambda\, d_A(a) d_A(b), \quad \forall a, b \in A.$$

Example

Let $A = C^\infty(\mathbb{R})$ be the algebra of smooth functions on $\mathbb{R}$. Let d be the classical derivation operation of smooth functions. Then A is a differential algebra of weight 0.

Part I: Zero weight vs nonzero weight

Remark

The defining relation of differential operators of any weight is given by

$$d_A \circ \mu_A = \mu_A \circ (d_A \otimes \mathrm{Id} + \mathrm{Id} \otimes \mu_A) + \lambda \ \mu_A \circ (d_A \otimes d_A)$$

expressed in terms of maps. If $\lambda = 0$, the operad of differential algebras of weight zero is Koszul, as shown by Loday in 2010.
When $\lambda \neq 0$, this relation is NOT quadratic and the operad of differential algebras of nonzero weight is not quadratic and not even homogeneous, so the Koszul duality theory for operads could not be applied directly to develop a cohomology theory of differential algebras of any weight.

Question (Loday)

If the parameter λ is different from 0, then the operad λ-AsDer is not a quadratic operad since the term $d(a)d(b)$ needs three generating operations to be defined. So one needs new techniques to extend Koszul duality to this case.

J.-L. Loday, *On the operad of associative algebras with derivation*, Georgian Math. J. **17** (2010), 347-372.

Part I: Non-Koszul case

Four steps:

$$\begin{array}{c}\text{formal deformations}\\ \Downarrow \\ \text{deformation complex}\\ \Downarrow \\ L_\infty\text{-structure}\\ \Updownarrow \\ \text{minimal model}\end{array}$$

Part II: Differential graded Lie algebras

Throughout this talk, let k be a field of characteristic zero.

Definition

A differential graded Lie algebra (aka dg Lie algebra) is a graded space $L = \oplus_{i \in \mathbb{Z}} L_i$ together with two operations:

$$l_1 : L_i \to L_{i-1}$$

of degree -1 and

$$l_2 : L_i \otimes L_j \to L_{i+j}$$

of degree zero such that

(i) $l_1 : L_i \to L_{i-1}$ *is a differential,*

(ii) $l_2 : L_i \otimes L_j \to L_{i+j}$ *is a Lie bracket,*

(iii) l_1 *is a derivation for* l_2 *,i.e.*

$$l_1 l_2(a \otimes b) = l_2(l_1(a) \otimes b) + (-1)^{|a|} l_2(a \otimes l_1(b))$$

for $a, b \in L$ homogeneous.

Part II: Maurer-Cartan elements in dg Lie algebras

Definition

Let L be a dg Lie algebra. An element $\alpha \in L_{-1}$ is a Maurer-Cartan element if

$$l_1(\alpha) - \frac{1}{2} l_2(\alpha \otimes \alpha) = 0.$$

Proposition (Twisting procedure)

Let L be a dg Lie algebra. Given a Maurer-Cartan element $\alpha \in L_{-1}$, one can produce a new dg Lie algebra by imposing

$$l_1^{\alpha}(x) = l_1(x) - l_2(\alpha \otimes x)$$

and

$$l_2^{\alpha}(x \otimes y) = l_2(x \otimes y)$$

Part II: L_∞-algebras

Definition

Let $L = \bigoplus_{i\in\mathbb{Z}} L_i$ *be a graded space over* k*. Assume that* L *is endowed with a family of linear operators* $l_n : L^{\otimes n} \to L, n \geq 1$ *with* $|l_n| = n-2$ *satisfying the following conditions:* $\forall \sigma \in S_n, x_1, \ldots, x_n \in L$,

(i) *(Skew-symmetry)*

$$l_n(x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(n)}) = \chi(\sigma, x_1, \ldots, x_n) l_n(x_1, \ldots, x_n),$$

(ii) *(Higher Jacobi identities)*

$$\sum_{i=1}^{n} \sum_{\sigma\in Sh(i,n-i)} \chi(\sigma, x_1, \ldots, x_n)(-1)^{i(n-i)}$$
$$l_{n-i+1}(l_i(x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(i)}) \otimes x_{\sigma(i+1)} \otimes \ldots \otimes x_{\sigma(n)}) = 0,$$

where $Sh(i, n-i)$ *is the set of* $(i, n-i)$ *shuffles, i.e.,* $Sh(i, n-i) = \{\sigma \in S_n$ *such that* $\sigma(1) < \sigma(2) < \cdots < \sigma(i),$ *and* $\sigma(i+1) < \sigma(i+2) < \ldots \sigma(n)\}$.

Then $(L, \{l_n\}_{n\geq 1})$ *is called a* L_∞*-algebra.*

Part II: Maurer-Cartan elements and L_∞-algebras

Definition

Let $(L, \{l_n\}_{n\geq 1})$ be an L_∞-algebra and $\alpha \in L_{-1}$. Then α is called a Maurer-Cartan element if it satisfies equation:

$$\sum_{n=1}^{\infty} \frac{1}{n!}(-1)^{\frac{n(n-1)}{2}} l_n(\alpha^{\otimes n}) = 0,$$

whenever this infinite sum exists.

Proposition

Let $(L, \{l_n\}_{n\geq 1})$ be an L_∞-algebra. Given a Maurer-Cartan element α in L_∞-algebra L, we can define a new L_∞ structure $\{l_n^\alpha\}_{n\geq 1}$ on graded space L, where $l_n^\alpha : L^{\otimes n} \to L$ is defined as :

$$l_n^\alpha(x_1 \otimes \ldots \otimes x_n) = \sum_{i=0}^{\infty} \frac{1}{i!}(-1)^{in+\frac{i(i-1)}{2}} l_{n+i}(\alpha^{\otimes i} \otimes x_1 \otimes \ldots \otimes x_n).$$

Part III: Differential algebras

Definition

Let $\lambda \in \mathbf{k}$ be a fixed element. A differential algebra of weight λ is an associative algebra (A, μ_A) together with a linear operator $d_A : A \to A$ such that

$$d_A(ab) = d_A(a)b + ad_A(b) + \lambda\, d_A(a)d_A(b), \quad \forall a, b \in A.$$

Remark

The defining relation of differential operators of any weight is given by

$$d_A \circ \mu_A = \mu_A \circ (d_A \otimes \mathrm{Id} + \mathrm{Id} \otimes \mu_A) + \lambda\, \mu_A \circ (d_A \otimes d_A)$$

expressed in terms of maps. If $\lambda = 0$, the operad of differential algebras of weight zero is Koszul, but while $\lambda \neq 0$, it is NOT Koszul.

Part III: Formal deformations of differential operators

Definition

Let (A, μ_A, d_A) be a differential algebra. A 1-parameterized family

$$d_t = \sum_{i=0}^{\infty} d_i t^i, \ \ d_i \in \mathrm{Hom}(A, A).$$

is called a **1-parameter formal deformation** *of the differential operators d_A if d_t is a differential operators of weight λ on the associative algebra $(A[[t]], \mu_A)$ such that $d_0 = d_A$.*

For all $x, y, z \in A$, the following equalities hold:

$$d_t(\mu_t(x, y)) \quad = \quad \mu_A(d_t(x), y) + \mu_A(x, d_t(y)) + \lambda\mu_A(d_t(x), d_t(y)).$$

Part III: Formal deformations of differential algebras

For any $n \geq 1$, we have

$$d_n \mu_A(x, y) = \mu_A(d_n(x), y) + \mu_A(x, d_n(y)) + \lambda \sum_{\substack{l, m \geq 0 \\ l+m=n}} \mu_A(d_l(x), d_m(y)).$$

Consider the case $n = 1$,

$$(x + \lambda d_A(x))d_1(y) - d_1(xy) + d_1(x)(y + \lambda d_A(y)) = 0. \tag{1}$$

Part III: New structures arising from a differential algebra

Let (A, μ_A, d_A) be a differential algebra of weight λ. Let M be a differential bimodule.

Proposition

We define a left action "$\triangleright$" and a right action "$\triangleleft$" of A on M as follows: for any $a \in A, x \in M$,

$$a \triangleright x := ax + \lambda d_A(a)x, \quad (2)$$

$$x \triangleleft a := xa + \lambda x d_A(a). \quad (3)$$

Then "$\triangleright$" and "$\triangleleft$" make M into a bimodule over A and denote this new bimodule by ${}_{\triangleright}M_{\triangleleft}$.

Deformation equation:

$$x \triangleright d_1(y) - d_1(xy) + d_1(x) \triangleleft y = 0 \Leftrightarrow \partial_{\mathrm{Alg}}(d_1) = 0 \in C^2_{\mathrm{Alg}}(A, {}_{\triangleright}A_{\triangleleft}). \quad (4)$$

Part III: Cohomology theory of differential operators

Definition

Let (A, μ_A, d_A) be a differential algebra of weight λ. Let M be a differential bimodule. Then the cochain complex

$$C^{\bullet}_{\mathrm{DO}_\lambda}(A, M) =:= (C^{\bullet}_{\mathrm{Alg}}(A, {}_{\triangleright}M_{\triangleleft}), \partial),$$

i.e, the Hochschild cochain complex of A with coefficients in ${}_{\triangleright}M_{\triangleleft}$, is called the cochain complex of differential operator d with coefficients in the differential bimodule M.

The cohomology groups of $C^{\bullet}_{\mathrm{DO}_\lambda}(A, \mu, d)$ are called the cohomology groups of differential operator d, denoted by $\mathrm{H}^{\bullet}_{\mathrm{DO}_\lambda}(A)$.

Part III: Formal deformations of differential algebras

Definition

Let (A, μ_A, d_A) be a differential algebra. A 1-parameterized family

$$\mu_t = \sum_{i=0}^{\infty} \mu_i t^i,\ \ \mu_i \in \mathrm{Hom}(A \otimes A, A), \quad d_t = \sum_{i=0}^{\infty} d_i t^i,\ \ d_i \in \mathrm{Hom}(A, A).$$

is called a **1-parameter formal deformation** *of the differential algebra (A, d_A) if the pair (μ_t, d_t) endows the $\mathbf{k}[[t]]$-module $(A[[t]], \mu_t, d_t)$ with a differential algebra structure over $\mathbf{k}[[t]]$ such that $(\mu_0, d_0) = (\mu_A, d_A)$.*

For all $x, y, z \in A$, the following equalities hold:

$$\begin{aligned} \mu_t(\mu_t(x, y), z) &= \mu_t(x, \mu_t(y, z)), \\ d_t(\mu_t(x, y)) &= \mu_t(d_t(x), y) + \mu_t(x, d_t(y)) + \lambda \mu_t(d_t(x), d_t(y)). \end{aligned}$$

Part III: Formal deformations of differential algebras

For any $n \geq 1$, we have

$$\sum_{i}^{n} \mu_i(\mu_{n-i}(x, y), z) = \sum_{i}^{n} \mu_i(x, \mu_{n-i}(y, z)),$$

$$\sum_{\substack{k,l \geq 0 \\ k+l=n}} d_l \mu_k(x, y) = \sum_{\substack{k,l \geq 0 \\ k+l=n}} (\mu_k(d_l(x), y) + \mu_k(x, d_l(y))) + \lambda \sum_{\substack{k,l,m \geq 0 \\ k+l+m=n}} \mu_k(d_l(x), d_m(y)).$$

Consider the case $n = 1$,

$$x\mu_1(y, z) - \mu_1(xy, z) + \mu_1(x, yz) - \mu_1(x, y)z = 0, \quad (5)$$

$$x \triangleright d_1(y) - d_1(xy) + d_1(x) \triangleleft y = d_A\mu_1(x, y) - \lambda\mu_1(d_A(x), d_A(y)). \quad (6)$$

Part III: A chain map

Define a chain map $\Phi^* : C^*_{\mathrm{Alg}}(A, M) \to C^*_{\mathrm{DO}_\lambda}(A, M)$ as follows: for any $f \in C^n_{\mathrm{Alg}}(A, M)$ with $n \geq 1$,

$$\begin{aligned} &\Phi^n(f)(a_{1,n}) \\ =&\sum_{k=1}^{n} \lambda^{k-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} f(a_{1,i_1-1}, d_A(a_{i_1}), a_{i_1+1,i_2-1}, d_A(a_{i_2}), \cdots, d_A(a_{i_k}), a_{i_k+1,n}) \\ &- d_M(f(a_{1,n})), \end{aligned}$$

and

$$\Phi^0(x) = -d_M(x), \quad \forall\, x \in C^0_{\mathrm{Alg}}(A, M) = M.$$

Part III: Return to formal deformations

Consider the case $n = 1$,

$$x\mu_1(y,z) - \mu_1(xy,z) + \mu_1(x,yz) - \mu_1(x,y)z = 0 \Longleftrightarrow \partial^2_{\mathrm{Alg}}\mu_1 = 0,$$

$$x \triangleright d_1(y) - d_1(xy) + d_1(x) \triangleleft y = d_A\mu_1(x,y) - \lambda\mu_1(d_A(x), d_A(y))$$

$$\Longleftrightarrow \partial^2_{\mathrm{DO}_\lambda} d_1 + \Phi^2\mu_1 = 0$$

Proposition

Case $n = 1 \Longleftrightarrow \partial^2_{\mathrm{DA}_\lambda}(\mu_1, d_1) = 0.$

Part III: Cohomology theory of differential algebras

Definition

The negative shift of the mapping cone of Φ^*, denoted by

$$(C^*_{\mathrm{DA}_\lambda}(A, M), \partial^*_{\mathrm{DA}_\lambda})$$

is called the cochain complex of the differential algebra A with coefficients in the differential bimodule M.

When the differential bimodule M is the regular differential bimodule A, we write $(C^*_{\mathrm{DA}_\lambda}(A), \partial^*_{\mathrm{DA}_\lambda}) := (C^*_{\mathrm{DA}_\lambda}(A, M), \partial^*_{\mathrm{DA}_\lambda})$.

Part III: L_∞-algebra structure on $\mathfrak{C}_{\mathrm{DA}_\lambda}(V)$

Let V be a graded space. Define a graded space $\mathfrak{C}_{\mathrm{DA}_\lambda}(V)$ as :

$$\mathfrak{C}_{\mathrm{DA}_\lambda}(V) = \mathfrak{C}_{\mathrm{Alg}}(V) \oplus \mathfrak{C}_{\mathrm{DO}_\lambda}(V),$$

where

$$\mathfrak{C}_{\mathrm{Alg}}(V) = \mathrm{Hom}(T^c(sV), sV), \mathfrak{C}_{\mathrm{DO}_\lambda}(V) = \mathrm{Hom}(T^c(sV), V).$$

Theorem

Given a graded space V and an element $\lambda \in \mathbf{k}$, there is an L_∞-algebra structure on graded space

$$\mathfrak{C}_{\mathrm{DA}_\lambda}(V)$$

such that it becomes Gerstenhaber graded Lie algebra over $\mathfrak{C}_{\mathrm{Alg}}(V)$.

Part III: L_∞-algebra structure on $\mathfrak{C}_{\mathrm{DA}_\lambda}(V)$

Theorem

Let V be an ungraded vector space.

- *The set of Maurer-Cartan elements in the L_∞-algebra $\mathfrak{C}_{\mathrm{DA}_\lambda}(V)$ is in bijection with the differential algebra structure of weight λ on V.*
- *Given a Maurer-Cartan element α in $\mathfrak{C}_{\mathrm{DA}_\lambda}(V)$, the underlying complex of the twisted L_∞-algebra by this Maurer-Cartan element is just the cochain complex of the corresponding differential algebra.*

Part III: Homotopy differential algebras

Definition

Let V be a graded space. A homotopy differential algebra structure of weight λ on V is defined to be a Maurer-Cartan element in the L_∞-algebra $\overline{\mathfrak{C}_{\mathrm{DA}_\lambda}}(V)$.

Part III: Homotopy differential algebras

Definition

Let V be a graded space. Then a homotopy differential algebra structure of weight λ on V consists of two families of operators $\{m_n\}_{n\geqslant 1}, \{d_n\}_{n\geqslant 1}$ with $m_n : V^{\otimes n} \to V, |m_n| = n-2, d_n : V^{\otimes n} \to V, |d_n| = n-1$ satisfying the following two conditions:

(1)

$$\sum_{1\leqslant i+j\leqslant n} (-1)^{i+jk} m_{n-j+1} \circ \left(\mathrm{id}^{\otimes i} \otimes m_j \otimes \mathrm{id}^{\otimes n-i-j}\right) = 0,$$

(2)

$$\sum_{i+j+k=n} (-1)^{i+jk} d_{n-j+1}(\mathrm{id}^{\otimes i} \otimes m_j \otimes \mathrm{id}^{\otimes k})$$

$$= \sum_{\substack{j_1+\cdots+j_{q+1}+q=p,\\ l_1+\cdots+l_q+j_1+\cdots+j_{q+1}=n}} (-1)^{\eta} \lambda^{q-1} m_p \circ (\mathrm{id}^{\otimes j_1} \otimes d_{l_1} \otimes \mathrm{id}^{\otimes j_2} \otimes \ldots \otimes d_{l_q} \otimes \mathrm{id}^{\otimes j_{q+1}}),$$

Part III: Homotopy differential algebras

(i) when $n = 1$, $|d_1| = 0$ and

$$d_1 \circ m_1 = m_1 \circ d_1,$$

that is, $d_1 : (V, m_1) \to (V, m_1)$ is a chain map;

(ii) when $n = 2$, $|d_2| = 1$ and and

$$d_1 \circ m_2 - \Big(m_2 \circ (d_1 \otimes \mathrm{Id}) + m_2 \circ (\mathrm{Id} \otimes d_1) + \lambda m_2 \circ (d_1 \otimes d_1) \Big) \\ = d_2 \circ (\mathrm{Id} \otimes m_1 + m_1 \otimes \mathrm{Id}) + m_1 \circ d_2,$$

which shows that d_1 is, up to a homotopy given by d_2, a differential operator of weight λ with respect to the "multiplication" m_2.

Part III: Minimal model

Theorem

The dg operad $_\lambda\mathfrak{Dif}_\infty$ *of homotopy differential algebras of weight* λ *is the minimal model of the operad* $_\lambda\mathfrak{Dif}$ *of differential algebras of weight* λ*, that is, there is a surjective quasi-isomorphism of operads* $p : {_\lambda\mathfrak{Dif}_\infty} \to {_\lambda\mathfrak{Dif}}$ *subject to a certain minimality condition.*

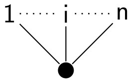
Remark

One can introduce the Koszul dual $_\lambda\mathfrak{Dif}^{¡}$ *such that* $\Omega({_\lambda\mathfrak{Dif}^{¡}}) = {_\lambda\mathfrak{Dif}_\infty}$*, but* $_\lambda\mathfrak{Dif}^{¡}$ *is only a homotopy cooperad.*

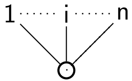
J. Chen, L. Guo, K. Wang and G. Zhou, *Koszul duality, minimal model and* L_∞*-structure for differential algebras with weight*. arXiv:2302.13216.

Part III: Minimal model

As a free graded operad, the dg operad ${}_\lambda\mathfrak{Dif}_\infty$ is generated by



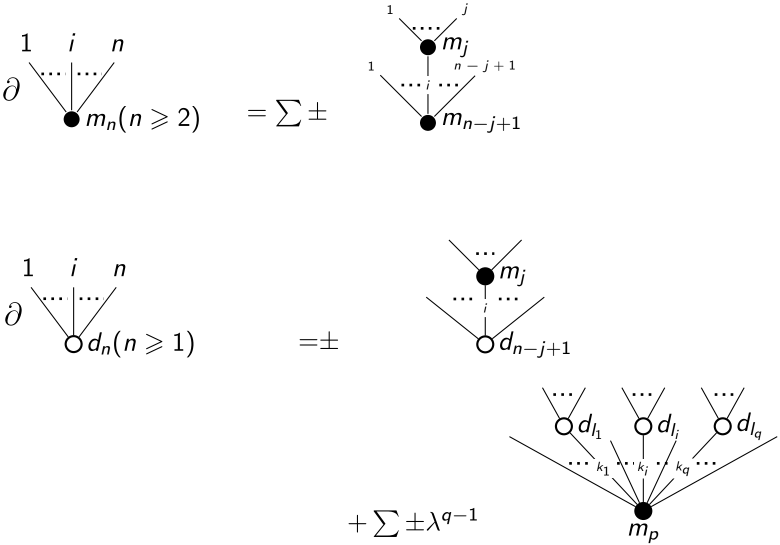
$m_n (n \geqslant 2)$



$d_n (n \geqslant 1)$

Part III: Minimal model

The differential is given by



Part IV: (Relative) Rota-Baxter associative/Lie algebras

Definition

Let $(R, \mu = \cdot)$ *be an associative algebra and* $\lambda \in k$*. A linear operator* $T : R \to R$ *is said to be a* **Rota-Baxter operator of weight** λ *if it satisfies*

$$\mu \circ (T \otimes T) = T \circ \mu \circ (Id \otimes T) + T \circ \mu \circ (T \otimes Id) + \lambda T \circ \mu. \tag{7}$$

Then (R, μ, T) *is called a* **Rota-Baxter algebra of weight** λ*.*

Definition

A relative Rota-Baxter Lie algebra is a triple $((\mathfrak{g}, [-,-]_{\mathfrak{g}}), \rho, T)$*, where* $(\mathfrak{g}, [-,-]_{\mathfrak{g}})$ *is a Lie algebra,* $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ *is a representation of* $\mathfrak{g}$ *on a vector space* V *and* $T : V \to \mathfrak{g}$ *is a relative Rota-Baxter operator, i.e.*

$$[Tu, Tv]_{\mathfrak{g}} = T(\rho(Tu)(v) - \rho(Tv)(u)), \forall u, v \in V$$

Part IV: (Relative) Rota-Baxter associative/Lie algebras

- Can define the deformation complex and construct the L_∞-structure
- Can define homotopy version

R. Tang, C. Bai, L. Guo and Y. Sheng, *Deformations and their controlling cohomologies of O-operators*. Comm. Math. Phys. **368** (2019), no. 2, 665-700.

A. Lazarev, Y. Sheng and R. Tang, *Deformations and Homotopy Theory of Relative Rota-Baxter Lie Algebras*. Comm. Math. Phys. **383** (2021), no. 1, 595-631.

A. Lazarev, Y. Sheng and R. Tang, *Homotopy relative Rota-Baxter Lie algebras, triangular L_∞-bialgebras and higher derived brackets*. Trans. Amer. Math. Soc. to appear.

K. Wang and G. Zhou, *Deformations and homotopy theory of Rota-Baxter algebras of any weight*. arXiv:2108.06744.

K. Wang and G. Zhou, *The homotopy theory and minimal model of Rota-Baxter algebras of arbitrary weight*. arXiv:2203.02960.

J, Chen, Z. Qi, K. Wang and G. Zhou, *The homotopy theory, minimal model and L_∞-structure of (relative) Rota-Baxter Lie*

Part IV: (Relative) Rota-Baxter associative/Lie algebras

- Can prove the minimal model

K. Wang and G. Zhou, *Deformations and homotopy theory of Rota-Baxter algebras of any weight*. arXiv:2108.06744.

K. Wang and G. Zhou, *The homotopy theory and minimal model of Rota-Baxter algebras of arbitrary weight*. arXiv:2203.02960.

J, Chen, Z. Qi, K. Wang and G. Zhou, *The homotopy theory, minimal model and* L_∞*-structure of (relative) Rota-Baxter Lie algebras of arbitrary weight*. in preparasion.

Part IV: Averaging algebras and embedding tensors

Definition

Let R be an associative algebra over field k. An averaging operator over R is a k-linear map $A : R \to R$ such that

$$A(x)A(y) = A(A(x)y) = A(xA(y))$$

for all $x, y \in R$.

Definition

Let $\mathfrak{g}$ be a Lie algebra over a field k and V a representation of $\mathfrak{g}$. A k-linear map $A : V \to \mathfrak{g}$ is an embedding tensor if

$$[A(x), A(y)] = A(A(x)y)$$

for all $x, y \in V$.

Part IV: Averaging algebras and embedding tensors

- Deformation complex, and L_∞-structure on deformation complex

K. Wang, G. Zhou, *Cohomology theory of averaging algebras, L_∞-structures and homotopy averaging algebras*, arXiv:2009.11618.

Y. Sheng, R. Tang and C. Zhu, *The controling L_∞-algebras, cohomology and homotopy of embedding tensors and Lie-Leibniz triples*, Comm. Math. Phys. **386** (2021), no. 1, 269-304

- Minimal model?

Part IV: Nijenhuis algebras

Definition

Let $(A, \mu = \cdot)$ be an associative algebra over field $\mathbf{k}$*. A linear operator* $P : A \to A$ *is said to be a Nijenhuis operator if it satisfies*

$$\mu \circ (P \otimes P) = P \circ \big(\mu \circ (\mathrm{Id} \otimes P) + \mu \circ (P \otimes \mathrm{Id}) - P \circ \mu\big). \tag{8}$$

In this case, (A, μ, P) is called a Nijenhuis algebra.

Theorem

- *Can define deformation complex of Nijenhuis algebras*
- *Can construct L_∞-structure on the deformation complex and define homotopy Nijenhuis algebras*
- *Can prove the minimal model*
- *Have some applications to Nijenhuis geometry*

C. Song, K. Wang, Y. Zhang, G. Zhou, *The homotopy theory of Nijenhuis algebras with geometric applications*, in preparasion.

A. V. Bolsinov, A. Yu. Konyaev, V. S. Matveev, *Nijenhuis geometry*, Adv. Math. **394**, (2022), 108001

Thank you very much!