Integration in cones

LHC days 2024

Thomas Ehrhard, IRIF, CNRS and Univ Paris Cité
Joint work with Guillaume Geoffroy, IRIF

June 6, 2024

Probabilities as an effect

In functional programming, probabilities are often considered as an *effect*.

Semantically (since Moggi): a CCC $\mathcal{C}$ together with a commutative strong monad $\mathcal{G}$.

Ideally: $\mathcal{C} = \mathbf{Meas}$, the category of measurable spaces and measurable functions + Giry monad... but **Meas** is not a CCC!

Solution: embed **Meas** in a CCC of (well-behaved) presheaves on **Meas** $\rightsquigarrow$ *Quasi Borel Spaces* (category **QBS**). There is a "Giry monad" $\mathcal{G} : \mathbf{QBS} \to \mathbf{QBS}$.

Probabilities as a non-effect

All the objects of the model have a basic *algebraic* probability structure, typically *convex spaces* with an operation of (sub-)barycentric combination.

Probabilistic Coherence Spaces (PCS) are a model of this kind (also a model of full classical LL). *Cones* are a generalization of PCS accounting for *continuous* probabilities.

Benefit

Morphisms are very regular (linear, analytic) because they must respect the probabilistic algebraic structure.

In **QBS** morphisms are only measurable, a very weak condition.

Question

Cones can be seen as QBS with a "generalized convex structure". Is **ICones** monadic over **QBS**?

The algebraic theory of cones

Selinger's cones

A cone is an $\mathbb{R}_{\geq 0}$ semi module P such that

- $x + y = 0 \Rightarrow x = 0$
- $x + y = x' + y \Rightarrow x = x'$

together with a norm $\|\_\| : P \to \mathbb{R}_{\geq 0}$

- $\|x\| = 0 \Rightarrow x = 0$
- $\|\lambda x\| = \lambda \|x\|$
- $\|x + y\| \leq \|x\| + \|y\|$
- positivity: $\|x\| \leq \|x + y\|$

Cone order and completeness

Definition

$x \leq y$ if there is z such that $x + z = y$

Then z is unique, notation: $y - x = z$

Fact

This is an order relation.

A cone must also satisfy

Any monotone $(x_n)_{n\in\mathbb{N}}$ with $\forall n \ \|x_n\| \leq 1$ has a lub $x = \sup_{n\in\mathbb{N}} x_n$ such that $\|x\| \leq 1$.

Main example

X a measurable space.

Then FMeas(X), the set of all nonnegative finite measures on X, is a cone with, for all $U \in \sigma_X$:

- $(\mu + \nu)(U) = \mu(U) + \nu(U)$, $(\lambda\,\mu)(U) = \lambda\,\mu(U)$
- so $\mu \leq \nu$ means $\forall U \in \sigma_X\ \mu(U) \leq \nu(U)$
- $\|\mu\| = \mu(X)$ (total variation norm).

Any probabilistic coherence space can be seen as a cone.

Our model of cones is a conservative extension of PCS.

Linear and continuous functions

Definition

$f : P \to Q$ is linear if $f(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 f(x_1) + \lambda_2 f(x_2)$

Fact

If f is linear then

- *f is increasing*
- *$f(\mathcal{B}P)$ is norm-bounded where $\mathcal{B}P = \{x \in P \mid \|x\| \leq 1\}$.*

Definition

f is *continuous* if $f(\sup_{n\in\mathbb{N}} x_n) = \sup_{n\in\mathbb{N}} f(x_n)$ for all monotone and bounded $(x_n \in P)_{n\in\mathbb{N}}$.

Much stronger than continuity for the topology induced by the norm.

The category of cones

If $f : P \to Q$ linear,

$$\|f\| = \sup_{x \in \mathcal{B}P} \|f(x)\| \in \mathbb{R}_{\geq 0}$$

Definition

Cones is the category of cones and linear and continuous functions f such that $\|f\| \leq 1$.

Example of morphism

$\kappa : X \rightsquigarrow Y$ means that κ is a *bounded kernel*, that is:

- $\kappa : X \to \mathrm{FMeas}(Y)$
- $\{\|\kappa(r)\| = \kappa(r)(Y) \mid r \in X\}$ is bounded in $\mathbb{R}_{\geq 0}$
- for all $V \in \sigma_Y$, $\kappa(\_)(V) : X \to \mathbb{R}_{\geq 0}$ is measurable.

Then κ induces a linear and continuous function

$$\widehat{\kappa} : \mathrm{FMeas}(X) \to \mathrm{FMeas}(Y)$$
$$\mu \mapsto \boldsymbol{\lambda} V \in \sigma_Y \cdot \int_{r \in X} \kappa(r)(V)\, \mu(dr)$$

Problem

There are a lot of linear continuous $f : \mathrm{FMeas}(X) \to \mathrm{FMeas}(Y)$ which are not induced by kernels.

And even worse...

If $f : \mathrm{FMeas}(X) \to \mathrm{FMeas}(Y)$ is linear and continuous the function

$$\begin{aligned} \kappa' : X &\to \mathrm{FMeas}(Y) \\ r &\mapsto f(\boldsymbol{\delta}^X(r)) \end{aligned}$$

is not necessarily a kernel: for $V \in \sigma_Y$, the function $f(\boldsymbol{\delta}^X(\_))(V) : X \to \mathbb{R}_{\geq 0}$ has no reason to be measurable.

Of course if $f = \widehat{\kappa}$ for a kernel κ then $\kappa' = \kappa$.

So we need...

... an additional structure on a cone P to enforce measurability.

Measurability

Measurability structure on a cone

Ar a *small* full subcategory of the category of measurable spaces and measurable functions which is closed by cartesian product. Typically $\mathbb{R} \in \mathbf{Ar}$. Terminal object 0.

A *measurability structure* on the cone P is a family $\mathcal{M} = (\mathcal{M}_X)_{X \in \mathbf{Ar}}$ where

$$\mathcal{M}_X \subseteq (\mathbb{R}_{\geq 0})^{X \times P}$$

is a collection of *measurability tests* on P.

- If $m \in \mathcal{M}_X$ and $r \in X$, $m(r, \_) \in \mathbf{Cones}(P, \mathbb{R}_{\geq 0})$;
- If $m \in \mathcal{M}_X$ and $x \in P$, $m(\_, x) : X \to \mathbb{R}_{\geq 0}$ is measurable;
- If $m \in \mathcal{M}_X$ and $\varphi \in \mathbf{Ar}(Y, X)$, then $m(\varphi(\_), \_) \in \mathcal{M}_Y$;
- If $x \neq y \in P$, there is $m \in \mathcal{M}_0$ such that $m(x) \neq m(y)$;
- $\|x\| = \sup\left\{\frac{m(x)}{\|m\|} \mid m \in \mathcal{M}_0 \setminus \{0\}\right\}$.

Example

$X \in \mathbf{Ar}$.

Given $Y \in \mathbf{Ar}$ and $U \in \sigma_X$, define

$$\begin{aligned} \widetilde{U} : Y \times \mathrm{FMeas}(X) &\to \mathbb{R}_{\geq 0} \\ (s, \mu) &\mapsto \mu(U) \end{aligned}$$

and $\mathcal{M}_Y = \{\widetilde{U} \mid U \in \sigma_Y\}$.

$\mathcal{M}$ is a measurability structure on $\mathrm{FMeas}(X)$.

Remark

These tests $\widetilde{U}$ do not use the parameter s in Y.

It will be used only by measurability tests on cones of functions.

Measurable paths

A measurable cone is a pair $B = (\underline{B}, \mathcal{M}^B)$ where

- $\underline{B}$ is a cone
- $\mathcal{M}^B$ is a measurability structure on $\underline{B}$.

Definition (measurable path in a measurable cone)

A *measurable path* from $X \in \mathbf{Ar}$ to B is a function $\beta : X \to \underline{B}$ which is bounded ($\beta(X)$ bounded in $\underline{B}$) and such that:

$$\forall Y \in \mathbf{Ar}\ \forall m \in \mathcal{M}^B_Y \quad \text{the map} \quad \begin{aligned} Y \times X &\to \mathbb{R}_{\geq 0} \\ (s, r) &\mapsto m(s, \beta(r)) \end{aligned}$$

is measurable.

They form a cone $\mathrm{Path}(X, B)$

Remark: $\underline{B}$, equipped with its measurable paths, is a QBS.

Lin., cont. and measurable functions

Definition (measurability of linear continuous maps)

Let B and C be measurable cones.

A linear and continuous function $f : \underline{B} \to \underline{C}$ is measurable if, for any B-measurable path $\beta : X \to \underline{B}$, the function $f \circ \beta$ is a C-measurable path $X \to \underline{C}$.

Example

We consider $\mathrm{FMeas}(X)$ as a measurable cone.

If $\kappa : X \rightsquigarrow Y$ then $\widehat{\kappa} : \underline{\mathrm{FMeas}(X)} \to \underline{\mathrm{FMeas}(Y)}$ is a linear, continuous and measurable functions.

But there are still linear, continuous and measurable $f : \underline{\mathrm{FMeas}(X)} \to \underline{\mathrm{FMeas}(Y)}$ which are not of shape $\widehat{\kappa}\ldots$

Integrals

Integral of a measurable path

Let $\beta : X \to \underline{B}$ be a measurable path and $\mu \in \underline{\mathrm{FMeas}(X)}$.

Definition (integral of a measurable path)

An integral of β wrt. μ is an $x \in \underline{B}$ such that

$$\forall m \in \mathcal{M}_0^B \quad m(x) = \int m(\beta(r))\, \mu(dr)\,,$$

a well defined Lebesgue integral $\in \mathbb{R}_{\geq 0}$ since $m \circ \beta : X \to \mathbb{R}_{\geq 0}$ is a bounded measurable function and μ is a finite measure.

If x exists, it is unique because $\mathcal{M}_0^B$ separates $\underline{B}$, notation:

$$x = \int \beta(r)\, \mu(dr)\,.$$

This is similar to the Pettis (aka. weak, Gelfand-Pettis) integral in topological vector spaces (1938).

For Banach space valued functions there is also a notion of *Bochner integral* based on integrals of simple functions like Lebesgue integrals, does not seem easily adaptable to cones however.

Integrable cone

Definition (integrable cone)

A measurable cone B is *integrable* if, for any $X \in \mathbf{Ar}$ and any $\mu \in \underline{\mathrm{FMeas}(X)}$, all the measurable paths $X \to \underline{B}$ have an integral wrt. μ.

NB: this is *property* of B, not a further structure.

Definition (integrable linear morphisms)

Let B and C be integrable cones. A linear and continuous $f : \underline{B} \to \underline{C}$ is *integrable* if it is measurable and, for any $\mu \in \underline{\mathrm{FMeas}(X)}$ and any measurable path $\beta : X \to \underline{B}$, one has

$$f\Big(\int \beta(r)\, \mu(dr)\Big) = \int f(\beta(r))\, \mu(dr)$$

Integration of measurable paths in integrable cones has all the good properties:

- linearity and commutation with lubs of monotone sequences
- Fubini theorem
- change of variable
- integrals with parameters
- etc

inherited from the standard Lebesgue integrals wrt. finite measures.

ICones the category whose objects are the integrable cones and

$$\mathbf{ICones}(B, C) = \{f : \mathbf{Cones}(\underline{B}, \underline{C}) \mid f \text{ meas. and integrable}\}.$$

Example

A measurable path $Y \to \underline{\mathrm{FMeas}(X)}$ is a finite kernel $\kappa : Y \rightsquigarrow X$.

If $\nu \in \underline{\mathrm{FMeas}(Y)}$, κ has an integral wrt. ν, namely

$$\widehat{\kappa}(\nu) = \boldsymbol{\lambda} U \in \sigma_X \cdot \int \kappa(s)(U)\, \nu(ds) \in \underline{\mathrm{FMeas}(X)}.$$

So $\mathrm{FMeas}(X)$ is an integrable cone.

In particular $\forall \mu \in \underline{\mathrm{FMeas}(X)} \quad \mu = \int \boldsymbol{\delta}^X(r)\mu(dr)$ since
$\left(\int \boldsymbol{\delta}^X(r)\mu(dr)\right)(U) = \int \boldsymbol{\delta}^X(r)(U)\mu(dr) = \int \chi_U(r)\mu(dr) = \mu(U).$

Dirac's are dense among finite measures

Let $f, g : \underline{\mathrm{FMeas}(X)} \to \underline{B}$ be linear continuous and integrable.

If $f(\boldsymbol{\delta}^X(r)) = g(\boldsymbol{\delta}^X(r))$ for all $r \in X$, then $f = g$:

if $\mu \in \underline{\mathrm{FMeas}(X)}$, we have

$$\begin{aligned} f(\mu) &= f\Big(\int \boldsymbol{\delta}^X(r)\,\mu(dr)\Big) \\ &= \int f(\boldsymbol{\delta}^X(r))\,\mu(dr) = \int g(\boldsymbol{\delta}^X(r))\,\mu(dr) = g(\mu) \end{aligned}$$

Consequence: if $f \in \underline{\mathrm{FMeas}(X)} \to \underline{\mathrm{FMeas}(Y)}$, there is $\kappa : X \rightsquigarrow Y$ such that $f = \widehat{\kappa}$, namely $\kappa = f \circ \boldsymbol{\delta}^X$.

Monoidal structure

Existence of left adjoints

ICones is locally small and complete, limits are computed as in **Set**.

$\mathbb{R}_{\geq 0}$ is cogenerating: if $f \neq g \in \textbf{ICones}(B, C)$ there is $h \in \textbf{ICones}(C, \mathbb{R}_{\geq 0})$ with $h\,f \neq h\,g$. Because $\mathcal{M}_0^B$ separates $\underline{B}$.

ICones is well-powered: the class of subobjects of any object is essentially small. Because **Ar** is small.

Consequence (by the SAFT)

For any locally small $\mathcal{C}$, any $F : \textbf{ICones} \to \mathcal{C}$ preserving all limits has a left adjoint.

Internal linear hom

If B and C are integrable cones, the set P of all linear, continuous, measurable and integrable functions $\underline{B} \to \underline{C}$ is a cone.

P has a measurability structure $\mathcal{M} = (\mathcal{M}_X)_{X \in \mathbf{Ar}}$ where

$$\mathcal{M}_X = \{\beta \triangleright p \mid \beta \in \underline{\mathrm{Path}(X, B)} \text{ and } p \in \mathcal{M}_X^C\}$$

$$\text{where} \quad \beta \triangleright p : X \times P \to \mathbb{R}_{\geq 0}$$
$$(r, f) \mapsto p(r, f(\beta(r)))$$

Fact

In that way one defines a measurable cone $B \multimap C$.

This measurable cone is integrable. The proof uses crucially the monotone convergence theorem.

Integrals are defined pointwise: if $\theta \in \underline{\mathrm{Path}(X, B \multimap C)}$ and $\mu \in \underline{\mathrm{FMeas}(X)}$, then

$$f = \int^{C \multimap D} \theta(r)\, \mu(dr) \in \underline{C \multimap D}$$

is given by

$$f(x) = \int^{D} \theta(r)(x)\, \mu(dr)$$

Fact

$\_ \multimap \_$ is a functor $\mathbf{ICones}^{\mathrm{op}} \times \mathbf{ICones} \to \mathbf{ICones}$ defined on morphisms by pre- and post-composition.

The tensor product

Fact

For each integrable cone, the functor $B \multimap \_ : \mathbf{ICones} \to \mathbf{ICones}$ *preserves all limits.*

So this functor has a left adjoint $\_ \otimes B : \mathbf{ICones} \to \mathbf{ICones}$.

Actually $\otimes : \mathbf{ICones} \times \mathbf{ICones} \to \mathbf{ICones}$ *is a functor (by functoriality of* $\_ \multimap \_$*).*

We have no explicit description of $A \otimes B$ yet.

ICones as a monoidal category

By adjunction we have natural bijection

$$\Phi_{A,B,C} : \mathbf{ICones}(A \otimes B, C) \to \mathbf{ICones}(A, B \multimap C)$$

Fact

We have

$$\Phi_{A,B,C} \in \mathbf{ICones}(A \otimes B \multimap C, A \multimap (B \multimap C))$$
$$\Phi^{-1}_{A,B,C} \in \mathbf{ICones}(A \multimap (B \multimap C), A \otimes B \multimap C)$$

$\rightsquigarrow$ monoidality isomorphisms.

Theorem

Equipped with $\otimes$ *and* $1 = \mathbb{R}_{\geq 0}$*, the category* **ICones** *is an SMCC.*

We already know that **ICones** is cartesian (it is complete).

Fact

$$\underline{\&_{i \in I} B_i} = \left\{ \overrightarrow{x} \in \prod_{i \in I} \underline{B_i} \mid (\|x_i\|_{B_i})_{i \in I} \text{ is bounded} \right\}$$

$$\left\| \overrightarrow{x} \right\| = \sup_{i \in I} \|x_i\|$$

Fact

ICones *is also cocomplete: this is another consequence of the SAFT. So we have cokernels etc., no clue about how to describe them concretely.*

Analytic functions

A function $f : \underline{B}^n \to \underline{C}$ is

- n-linear and continuous if it is so, separately in each argument;
- measurable if for all family $(\beta_i : X \to \underline{B})_{i=1}^n$ of measurable B-paths, $f \circ \langle \beta_1, \ldots, \beta_n \rangle$ is a measurable C-path;
- integrable if it commutes with integrals separately in each argument.

A function $h : \underline{B} \to \underline{C}$ is *n-homogeneous polynomial* if there is an $f : \underline{B}^n \to \underline{C}$ which is n-linear, continuous, measurable and integrable such that $h(x) = f(x, \ldots, x)$.

Fact

There is only one such symmetric f, obtained from h by polarization.

A function $f : \mathcal{B}\underline{B} \to \underline{C}$ is analytic if it is bounded and there is a family $(f_n)_{n\in\mathbb{N}}$ such that $f_n : \underline{B} \to \underline{C}$ is n-homogeneous and

$$\forall x \in \mathcal{B}B \quad f(x) = \sum_{n\in\mathbb{N}} f_n(x) = \sup_{N\in\mathbb{N}} \sum_{n=0}^{N} f_n(x)$$

Example

$f : [0,1] = \mathcal{B}\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ given by $f(x) = 1 - \sqrt{1-x}$.

Cannot be extended beyond $[0,1]$.

The $(f_n)_{n\in\mathbb{N}}$ is unique: for all $m \in \mathcal{M}_0^B$,

$$m(f_n(x)) = \frac{1}{n!}\frac{d^n}{dt^n}\, m(f(tx))\,|_{t=0}$$

Taylor expansion

So when $f : \mathcal{B}\underline{B} \to \underline{C}$, there are uniquely determined symmetric multilinear continuous and integrable functions

$$D^n f(0) : \underline{B}^n \to \underline{C}$$

(the derivatives of f at 0) such that

$$\forall x \in \mathcal{B}\underline{B} \quad f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n f(0) \cdot (x, \ldots, x)$$

Any analytic function $f : \mathcal{B}\underline{B} \to \underline{C}$ is increasing and Scott continuous: if $(x_n \in \mathcal{B}B)_{n\in\mathbb{N}}$ is monotone then

$$f(\sup_{n\in\mathbb{N}} x_n) = \sup_{n\in\mathbb{N}} f(x_n)$$

f is also measurable in the sense that $f \circ \beta$ is a C-measurable path for any B-measurable path β.

The analytic functions $f : \mathcal{B}\underline{B} \to \underline{C}$ form an integrable cone (norm, measurability structure and integration as in $B \multimap C$):

$$B \Rightarrow_{\mathrm{a}} C$$

The category of analytic functions

ACones$(B, C) = \mathcal{B}\underline{B \Rightarrow_{\mathrm{a}} C}$, composition as in **Set**.

Theorem

ACones *is a CCC.*

For any integrable cone, there is an analytic least fixpoint operator $\mathcal{Y} \in$ **ACones**$(B \Rightarrow_{\mathrm{a}} B, B)$ *such that*

$$\forall f \in \mathcal{B}\underline{B \Rightarrow_{\mathrm{a}} B} \quad \mathcal{Y}(f) = \sup_{n \in \mathbb{N}} f^n(0)$$

so that $\mathcal{Y}(f)$ *is the least fixpoint of* f*.*

The analytic exponential

There is a functor $\mathrm{Der} : \mathbf{ICones} \to \mathbf{ACones}$ such that

$$\mathrm{Der}(B) = B \quad \text{and} \quad \mathrm{Der}(f) = f$$

since $f \in \mathbf{ICones}(B, C) \Rightarrow f \in \mathbf{ACones}(B, C)$ (more precisely, the restriction of f to $\mathcal{B}\underline{B}$).

The functor Der preserves all limits.

So it has a left adjoint $\mathrm{E} : \mathbf{ACones} \to \mathbf{ICones}$.

$$\Psi_{B,C} : \mathbf{ICones}(\mathrm{E}(B), C) \simeq \mathbf{ACones}(B, C)$$

which induces a comonad $! = \mathrm{E} \circ \mathrm{Der} : \mathbf{ICones} \to \mathbf{ICones}$ with counit der and comultiplication dig, and

$$\mathbf{ICones}_! \simeq \mathbf{ACones}$$

Promotion

We have

$$\mathrm{an} = \Psi_{B,\mathrm{E}(B)}(Id) \in \mathbf{ACones}(B, !B)$$

the *universal analytic function*: for any $f \in \mathbf{ACones}(B, C)$, there is exactly one $g \in \mathbf{ICones}(!B, C)$ such that

$$f = g \circ \mathrm{an}$$

namely $g = \Psi_{B,C}^{-1}(f)$.

For $x \in \mathcal{B}\underline{B}$, we set $x^! = \mathrm{an}(x) \in \mathcal{B}\underline{!B}$.

Remark

If $f, g \in \mathbf{ICones}(!B, C)$ satisfy $f(x^!) = g(x^!)$ for all $x \in \mathcal{B}\underline{B}$, then $f = g$.

Integration, data-types and sampling

Cones of finite measures as data-types

Let $X \in \mathbf{Ar}$.

We define $h_X \in \mathbf{ICones}(\mathrm{FMeas}(X), !\mathrm{FMeas}(X))$ by

$$h_X(\mu) = \int (\boldsymbol{\delta}^X(r))^! \, \mu(dr)$$

$$X \xrightarrow{\boldsymbol{\delta}^X} \mathrm{FMeas}(X) \xrightarrow{\mathrm{an}} !\mathrm{FMeas}(X) \quad \text{measurable path.}$$

Fact

$(\mathrm{FMeas}(X), h_X)$ *is a* $!$*-coalgebra.*

We have used the fact that integration is possible in $!\mathrm{FMeas}(X)$, a major outcome of this approach!

Moreover for $\varphi \in \mathbf{Ar}(X, Y)$, we have the push-foward

$$\varphi_* : \underline{\mathrm{FMeas}(X)} \to \underline{\mathrm{FMeas}(Y)}$$
$$\mu \mapsto \boldsymbol{\lambda} V \in \sigma_Y \cdot \mu(\varphi^{-1}(V))$$

Fact (**Ar** is a full subcategory of $\mathbf{ICones}^!$)

$\varphi_* \in \mathbf{ICones}^!((\mathrm{FMeas}(X), h_X), (\mathrm{FMeas}(Y), h_Y)))$

So we have a functor $\mathbf{Ar} \to \mathbf{ICones}^!$ *which is easily seen to be faithful.*

If all the objects or **Ar** *are standard Borel spaces, then this functor is also full.*

NB: discrete $\mathbb{N}$, $\mathbb{R}$, Cantor space etc. are standard Borel spaces (= Polish spaces equipped with their Borel σ-algebra).

Polish space = complete metric space which has a countable dense subset.

How to interpret sampling

Imagine we have a programming language with types

$$\sigma, \tau, \cdots := \rho \mid \sigma \Rightarrow \tau \mid \cdots$$

where ρ is the type of real numbers. We choose **Ar** with $\mathbb{R} \in \mathbf{Ar}$.

$[\![\sigma]\!]$ is a measurable cone, $[\![\rho]\!] = \mathrm{FMeas}(\mathbb{R})$,
$[\![\sigma \Rightarrow \tau]\!] = [\![\sigma]\!] \Rightarrow_{\mathsf{a}} [\![\tau]\!]$

If $\vdash M : \rho$ then $[\![M]\!] \in \mathcal{B}\underline{\mathrm{FMeas}(\mathbb{R})}$.

If $x : \rho \vdash N : \sigma$ then $[\![N]\!] \in \mathbf{ACones}([\![\rho]\!], [\![\sigma]\!])$.

Then we can sample a real number according to the subdistribution M in N

$$\vdash \mathtt{sample}(x, M, N) : \sigma$$

We have $[\![N]\!]_{x:\rho} \in \mathbf{ICones}(!\mathsf{FMeas}(\mathbb{R}), [\![\sigma]\!])$ hence

$$[\![N]\!]_{x:\rho}\, h_{\mathsf{FMeas}(\mathbb{R})} \in \mathbf{ICones}(\mathsf{FMeas}(\mathbb{R}), [\![\sigma]\!])$$

and we take

$$[\![\mathtt{sample}(x, M, N)]\!] = [\![N]\!]_{x:\rho}(h_{\mathsf{FMeas}(\mathbb{R})}([\![M]\!]))$$

that is, considering $[\![N]\!]_{x:\rho}$ as an analytic function $\mathcal{B}\underline{\mathsf{FMeas}(\mathbb{R})} \to \underline{[\![\sigma]\!]}$,

$$\begin{aligned}[\![\mathtt{sample}(x, M, N)]\!] &= \int [\![N]\!]_{x:\rho}(\boldsymbol{\delta}^{\mathbb{R}}(r))\, [\![M]\!](dr) \\ &\neq [\![N]\!]_{x:\rho}([\![M]\!]) \quad \text{in general.}\end{aligned}$$

Sampling is just a `let` construct allowing to use the type ρ in call-by-value.