Polynomials in homotopy type theory

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École Polytechnique

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Overview

The situation:

- Polynomials in a category are a categorification of ordinary polynomials

$$F(X) = X \times X + 1$$

- They can be defined in any locally cartesian closed category
- Similar to combinatorial species and their generalizations

Our contribution:

- Show that polynomials are Kleisli morphisms for a comonad on spans
- and fit in a model of linear logic
- in order for this to work, we have to work up to homotopy!

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Semantics of linear logic

- Decomposing $(A \implies B)$ as $(!A \multimap B)$.
- e.g. $\mathrm{Poly}(\mathbb{R}^m, \mathbb{R}^n) \simeq \mathrm{Lin}(\mathbb{R}[X_1, \ldots, X_m], \mathbb{R}^n) \simeq \mathrm{Lin}(\mathrm{Sym}(\mathbb{R}^m), \mathbb{R}^n)$
- in categorical models: ! is a comonad on a symmetric monoidal category, satisfying some conditions
- e.g. $\mathbb{R}$-vector spaces, $! = \mathrm{Sym} =$ free symmetric algebra
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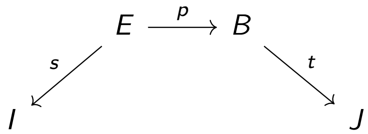
A polynomial from I to J in a category $\mathcal{C}$ is a diagram



When $\mathcal{C} = \mathrm{Set}$, it induces a **polynomial functor**

$$\mathrm{Set}^I \to \mathrm{Set}^J$$
$$(X_i)_{i \in I} \mapsto \Big(\sum_{b \in t^{-1}(j)} \prod_{e \in p^{-1}(b)} X_{s(e)} \Big)_{j \in J}$$

- "B = monomials"
- "E = exponents/arities"

Polynomials in categories

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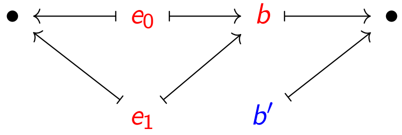
Example

$$\{\bullet\} \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} \{\bullet\}$$



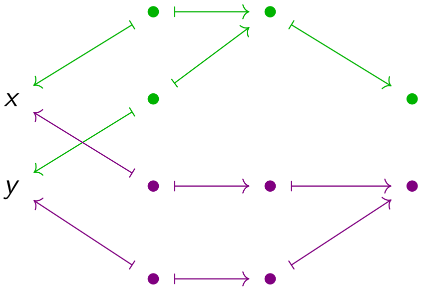
Induced functor:

$$\mathrm{Set} \to \mathrm{Set}$$

$$X \mapsto X^2 + 1$$

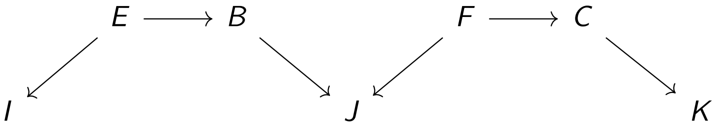
A more complicated example

$$\{0,1\} \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} \{0,1\}$$



$$\mathrm{Set}^2 \to \mathrm{Set}^2$$
$$(X, Y) \mapsto (XY, X + Y)$$

Composition of polynomials



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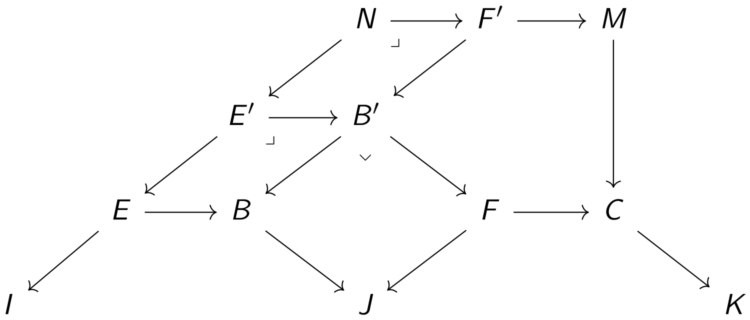

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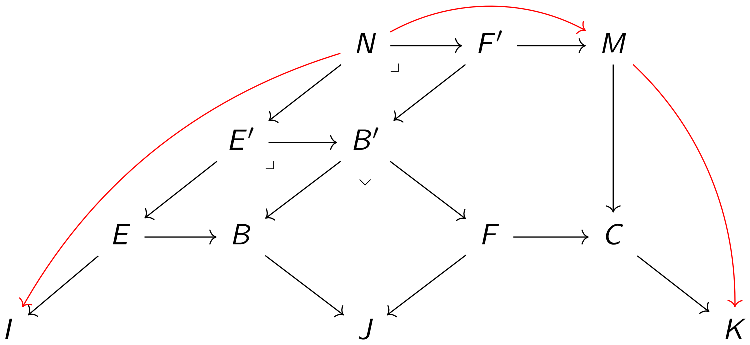

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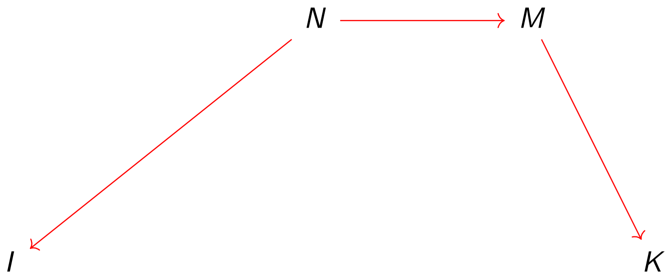

Linear polynomials: spans

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} J$$

is linear when p is an isomorphism: the products are taken over singletons

Linear polynomials are isomorphic to spans

And they compose via pullbacks

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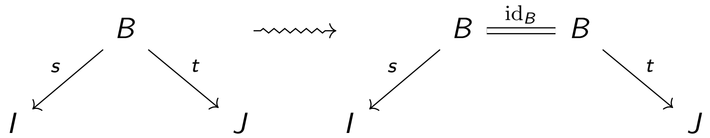

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$$\begin{array}{ccc} & B & \\ {\scriptstyle s}\swarrow & & \searrow{\scriptstyle t} \\ I & & J \end{array} \quad \rightsquigarrow \quad \begin{array}{cccc} & B \overset{\mathrm{id}_B}{=\!=\!=} B & & \\ {\scriptstyle s}\swarrow & & \searrow{\scriptstyle t} & \\ I & & & J \end{array}$$

And they compose via pullbacks

$$\begin{array}{ccccccc} & & & B \times_J C & & & \\ & & \swarrow & \lrcorner & \searrow & & \\ & B & & & & C & \\ {\scriptstyle s}\swarrow & & \searrow{\scriptstyle t} & & {\scriptstyle s'}\swarrow & & \searrow{\scriptstyle t'} \\ I & & & J & & & K \end{array}$$

The bicategory of polynomials

- Composition of polynomials uses pullbacks and other universal properties
- So it is not strictly associative/unital
- Polynomials thus form a **bicategory**
- But we need to further restrict it...

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- $\mathrm{Poly}(\mathbb{R}^m, \mathbb{R}^n) \simeq \mathrm{Lin}(\mathbb{R}[X_1, \ldots, X_m], \mathbb{R}^n)$
- A basis of $\mathbb{R}[X_1, \ldots, X_m]$ is given by monomials.
- In our categorified setting, we have arbitrary sets as exponents.
- So we would get a **proper class** of monomials !

Thus we restrict ourselves to **finitary polynomials**.

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Finitary polynomials

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} J$$

is said to be **finitary** if $\forall b \in B, p^{-1}(b)$ is finite.

Examples:

- $X \mapsto X^3 + X + 1$ is finitary
- $X \mapsto \mathbb{N} \times X$ is finitary
- $(X_i)_{i\in\mathbb{N}} \mapsto ((X_i)^i)_{i\in\mathbb{N}}$ is finitary
- $X \mapsto X^{\mathbb{N}}$ is **not** finitary
- a linear polynomial is always finitary

(more generally, we could restrict to polynomials with arities in a fixed universe $\mathcal{V}$ of "small" sets/types)

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Spans and polynomials

Span and Poly the bicategories of spans and finitary polynomials in sets.

- We would like $\mathrm{Poly}(I, J) \simeq \mathrm{Span}(!I, J)$
- Idea: monomials in $\mathbb{R}[X_1, \ldots, X_m]$ are given by **multisets** over $\{1, \ldots, n\}$
- Can we hope for $\mathrm{Poly}(I, J) \simeq \mathrm{Span}(\mathrm{Mul}(I), J)$?

Yes and no... At the level of sets, yes, but Poly and Span are bicategories, and those groupoids of morphisms are not equivalent !

We need to :

- replace multisets by **homotopy multisets**,
- replace sets by groupoids.

To do that, we work in Homotopy Type Theory.

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Homotopy Type Theory

- HoTT is an extension of Martin Löf Type Theory where types are thought of as **spaces**.
- Spaces in the sense of **homotopy theory**.
- Discrete types are sets.
- Groupoids also are types, as are 2-groupoids, n-groupoids, ∞-groupoids.
- In this context, Σ-types look like a Grothendieck construction.
- Quotients are **homotopy quotients** : instead of identifying elements, they paths between them.

$$\mathrm{Bool} \underset{\neg}{\overset{\mathrm{id}}{\rightrightarrows}} \mathrm{Bool} \longrightarrow \{\bullet\}$$ coequalizer

$$\mathrm{Bool} \underset{\neg}{\overset{\mathrm{id}}{\rightrightarrows}} \mathrm{Bool} \longrightarrow S^1$$ homotopy coequalizer

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$$\mathrm{Bool} \underset{\neg}{\overset{\mathrm{id}}{\rightrightarrows}} \mathrm{Bool} \longrightarrow \{\bullet\} \qquad \text{coequalizer}$$

$$\mathrm{Bool} \underset{\neg}{\overset{\mathrm{id}}{\rightrightarrows}} \mathrm{Bool} \longrightarrow S^1 \qquad \text{homotopy coequalizer}$$

Homotopy Type Theory

- HoTT is an extension of Martin Löf Type Theory where types are thought of as **spaces**.
- Spaces in the sense of **homotopy theory**.
- Discrete types are sets.
- Groupoids also are types, as are 2-groupoids, n-groupoids, ∞-groupoids.
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Spans and polynomials in HoTT

- Ordinary multisets: $\mathrm{Mul}(X) := \sum_{n:\mathbb{N}} X^n/\Sigma_n$
- Equivalently, the **free commutative monoid on** X.
- Homotopy multisets: $\mathrm{HMul}(X) := \sum_{n:\mathbb{N}} X^n /\!\!/ \Sigma_n$ (homotopy quotient).
- In category theory, $\mathrm{HMul}(X)$ is equivalently the **free symmetric monoidal groupoid on** X.
- Concretely in HoTT: $\mathrm{HMul}(X) := \sum_{E:\mathrm{Fin}} X^E$ where Fin is the **groupoid** of finite sets and bijections

With this last definition, and using spans and polynomials in **types**, we proved **in HoTT**:

$$\mathrm{Poly}(I, J) \simeq \mathrm{Span}(\mathrm{HMul}(I), J)$$

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A higher categorical model

- Switching from sets to groupoids makes Poly and Span into 3-categories.
- Going to arbitrary types, we get an ∞-category : associativity and unitality up to isomorphisms, themselves satisfying coherence laws, etc.
- We cannot state or prove those infinite **homotopy coherence laws** in HoTT, so we work with **wild categories**.
- Wild categories have the standard definition of categories, but with sets replaced by types.
- No pentagon or triangle isomorphisms required of the associators and unitors.

Remark

Not all coherences can be stated in HoTT, but some can be proven meta-theoretically. For instance, we can prove a wild category has cartesian products, and know meta-theoretically that the induced monoidal structure is homotopy coherent.

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A full model of classical linear logic

- Poly is the Kleisli category of a comonad HMul on Span.
- HMul makes Span into a Seely category: a model of intuitionistic linear logic.
- Span is moreover compact closed, with self-dual objects.
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Examples of higher polynomials - 1

$$\{\bullet\} \longleftarrow \mathbb{N} \xrightarrow{p} \sum_{n:\mathbb{N}} B(\mathbb{Z}/n\mathbb{Z}) \longrightarrow \{\bullet\}$$

$$n \longmapsto \{1, \ldots, n\}$$

- $B(\mathbb{Z}/n\mathbb{Z})$ is the groupoid with one point and $\mathbb{Z}/n\mathbb{Z}$ as automorphisms
- $p^{-1}(\{1, \ldots, n\}) \simeq \mathbb{Z}/n\mathbb{Z}$
- p^{-1} is taken in the sense of **homotopy fiber**
- Induced polynomial : $F(X) = \sum_{n:\mathbb{N}} X^n /\!\!/ (\mathbb{Z}/n\mathbb{Z})$
- The type of cyclic lists over X
- Generally, summing over groupoids amounts to quotienting the summand

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$$\{\bullet\} \longleftarrow \mathrm{Fin}_* \xrightarrow{p} \mathrm{Fin} \longrightarrow \{\bullet\}$$

$$(E, e) \longmapsto E$$

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Examples of higher polynomials - The Hopf Fibration

$$\{\bullet\} \longleftarrow S^3 \xrightarrow{h} S^2 \longrightarrow \{\bullet\}$$

- The map H has fiber $h(x)$, merely equivalent to S^1, the circle.
- $F(X) = \sum_{x:S^2} X^{h(x)}$.
- This locally looks like $S^2 \times X^{S^1}$, but in a globally twisted way.
- If you have any idea what this represents, please reach out !

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Conclusion/future work

We arranged the usual notions of spans and polynomials into a model of linear logic, using ideas from homotopy type theory. What next?

- Differential structure?
- Exploring other "homotopifications" of vector spaces and polynomials: spectra? stable ∞-categories?
- Comparison with other span-based models of linear logic by Mellies, Clairambault, Forest
- Comparison with generalized species of structure