What is an OGS?

Operational Game Semantics:

a semantic based on the analysis of normal forms to deduce the interactions with the environment.
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Operational Game Semantics: a semantic based on the analysis of normal forms to deduce the interactions with the environment.

Already a close composition: $\llbracket E[T] \rrbracket = \llbracket E \rrbracket \circ \llbracket T \rrbracket$.

Towards the definition of an open composition: $\llbracket TU \rrbracket = \llbracket T \rrbracket \circ \llbracket U \rrbracket$. 
A semantic category $C_{sem}$: based on LTSs and parallel composition;

A syntactic category $C_{syn}$: based on name assignations and the substitution;

A functor between $C_{syn}$ and $C_{sem}$. 
The $\lambda$-Calculus

Terms: $T, U \triangleq x | \lambda x.T | TU$

Towards Categorical Structures for Operational Game Semantics
The $\lambda$-Calculus

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Definition (Reduction)

Evaluation Contexts: $E, F \triangleq [.] | ET$

Reduction: $E[(\lambda x.T)U] \rightarrow E[T\{U/x\}]$

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The $\lambda$-Calculus

Terms: $T, U \triangleq x | \lambda x.T | TU$

Definition (Reduction)

Evaluation Contexts: $E, F \triangleq [\cdot] | ET$

Reduction: $E[(\lambda x.T)U] \rightarrow E[T\{U/x\}]$

$I = \lambda x.x$

$Ix \rightarrow_\beta x$

$\Omega = (\lambda x.xx)(\lambda x.xx)$

$\Omega \rightarrow_\beta \Omega \rightarrow_\beta \cdots \rightarrow_\beta \Omega \rightarrow_\beta \cdots$

$X = (\lambda x.(xx)f)(\lambda x.(xx)f)$

$X \rightarrow (((\lambda x.(xx)f)(\lambda x.(xx)f))f)$

$\cdots \rightarrow (((\cdots f)f)f)$

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**Definition (Game bipartite LTS $\mathcal{G} = (\text{Pos}, \text{Moves}, \to)$)**

With $\phi^\text{in}$ and $\phi^\text{in}$ sets of names $a, b$

\[
\frac{a \in \phi^\text{out} \quad b_1, \ldots, b_k \notin \phi^\text{in}}{\langle \phi^\text{in} \mid \phi^\text{out} \rangle \oplus \ a!(b_1, \ldots, b_k) \quad \langle \phi^\text{in} \cup \{b_1, \ldots, b_k\} \mid \phi^\text{out} \rangle \ominus}
\]

\[
\frac{a \in \phi^\text{in} \quad b_1, \ldots, b_k \notin \phi^\text{out}}{\langle \phi^\text{in} \mid \phi^\text{out} \rangle \ominus \ a?(b_1, \ldots, b_k) \quad \langle \phi^\text{in} \mid \phi^\text{out} \cup \{b_1, \ldots, b_k\} \rangle \oplus}
\]
A Few Definitions

- An **LTS morphism** from $\mathcal{L}_1 = (\text{STATES}_1, \text{ACTIONS}, \rightarrow_1)$ to $\mathcal{L}_2 = (\text{STATES}_2, \text{ACTIONS}, \rightarrow_2)$ is a function $f : \text{STATES}_1 \rightarrow \text{STATES}_2$ such that for all transitions $S \xrightarrow{\text{act}}_1 R$ of $\mathcal{L}_1$, there is $f(S) \xrightarrow{\text{act}}_2 f(R)$ in $\mathcal{L}_2$. 
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- A **game-indexed LTS** is a pair $(\mathcal{L}, \varnothing)$ formed by a bipartite LTS, together a bipartite LTS morphism $\varnothing$ between $\mathcal{L}$ and the Game LTS $\mathcal{G}$. 


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- A **game-indexed LTS** is a pair $(\mathcal{L}, \circ)$ formed by a bipartite LTS, together a bipartite LTS morphism $\circ$ between $\mathcal{L}$ and the Game LTS $\mathcal{G}$.

- A **strategy** $S \in \text{Strats}$ is a triple $(\mathcal{L}, \circ, S)$ formed by a game-indexed LTS $(\mathcal{L}, \circ)$, and a passive state $S$. We write $\text{Strats}[\mathcal{P}]$ for the strategies $(\mathcal{L}, \circ, S)$ such that $S \circ \mathcal{P}$. 

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A Few Definitions

- An **LTS morphism** from \( \mathcal{L}_1 = (\text{States}_1, \text{Actions}, \rightarrow_1) \) to \( \mathcal{L}_2 = (\text{States}_2, \text{Actions}, \rightarrow_2) \) is a function \( f : \text{States}_1 \rightarrow \text{States}_2 \) such that for all transitions \( S \xrightarrow{\text{act}}_1 R \) of \( \mathcal{L}_1 \), there is \( f(S) \xrightarrow{\text{act}}_2 f(R) \) in \( \mathcal{L}_2 \).

- A **game-indexed LTS** is a pair \((\mathcal{L}, \circ)\) formed by a bipartite LTS, together a bipartite LTS morphism \( \circ \) between \( \mathcal{L} \) and the Game LTS \( \mathcal{G} \).

- A **strategy** \( S \in \text{Strats} \) is a triple \((\mathcal{L}, \circ, S)\) formed by a game-indexed LTS \((\mathcal{L}, \circ)\), and a passive state \( S \).

We write \( \text{Strats}[P] \) for the strategies \((\mathcal{L}, \circ, S)\) such that \( S \circ P \).

- A game-indexed LTS \((\mathcal{L}, \circ)\) is **receptive** when for all \( S \circ P \) with \( P \) passive, if \( P \xrightarrow{m} Q \) then there exists a state \( R \) such that \( S \xrightarrow{m} R \) and \( R \circ Q \).
Parallel Composition $\mathcal{L}_P \parallel \mathcal{L}_O$

- states: $S_P H_{PO} \parallel H_{OP} S_O$ with $H_{PO}, H_{OP}$ hidden names,
- visible actions: Moves, silent actions: sync,
- transition function: (with $m = a(b_1, \ldots, b_k)$)

$$
\begin{align*}
S_O \xrightarrow{m} & \text{OR} \quad \text{SP passive} \quad a \not\in H \\
\frac{S_P H_{PO} \parallel H_{OP} S_O \xrightarrow{m}}{S_P H_{PO} \parallel H_{OP} R_O} \\
\frac{S_P \xrightarrow{m} \text{RP} \quad S_O \text{ passive} \quad a \not\in H}{S_P H_{PO} \parallel H_{OP} S_O \xrightarrow{m}} \\
\frac{S_P \xrightarrow{m} \text{RP} \quad S_O \xrightarrow{\bar{m}} \text{OR} \quad a \in H_{PO}}{S_P H_{PO} \parallel H_{OP} S_O \xrightarrow{\text{sync}}} \\
\frac{S_P H_{PO} \parallel H_{OP} S_O \xrightarrow{\text{sync}} \text{RP} \{b_1, \ldots, b_k\} \cup H_{PO} \parallel H_{OP} \text{RO}}{S_P H_{PO} \parallel H_{OP} S_O \xrightarrow{\text{sync}}} \\
\end{align*}
$$
There is an LTS morphism between $G||G$ and $G$. 

Definition ($C_{\text{sem}}$): objects: set of names $\phi$, morphisms between $\phi$ and $\psi$: strategies $S \in \text{Strats}[\langle \phi | \psi \rangle]$ quotiented by bisimilarity, composition of two morphisms as above, identity morphism over $\phi$: the bisimilarity quotient of the Forwarder strategy $F_\phi$. 

Towards Categorical Structures for Operational Game Semantics
There is an LTS morphism between $G \parallel G$ and $G$.

**Composition:** for $S_1 \in \text{Strats}[\langle \phi^{\text{in}} \mid \phi \rangle^\Theta]$ and $S_2 \in \text{Strats}[\langle \phi \mid \phi^{\text{out}} \rangle^\Theta]$

$$S_2 \circ S_1 = S_1 \phi \parallel \emptyset S_2 \in \text{Strats}[\langle \phi^{\text{in}} \mid \phi^{\text{out}} \rangle^\Theta].$$
There is an LTS morphism between $G || G$ and $G$.

**Composition:** for $S_1 \in \text{Strats}[\langle \phi \mid \phi \rangle^\Theta]$ and $S_2 \in \text{Strats}[\langle \phi \mid \phi^\text{out} \rangle^\Theta]$

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**Definition ($C_{sem}$)**

- objects: set of names $\phi$,

- morphisms between $\phi$ and $\psi$: strategies $S \in \text{Strats}[\langle \phi \mid \psi \rangle^\Theta]$ quotiented by bisimilarity,

- composition of two morphisms as above,

- identity morphism over $\phi$: the bisimilarity quotient of the Forwarder strategy $\mathcal{F}_\phi$. 
Names: $a = x \mid v \mid c$
Name assignations: partial maps s.t. $\gamma(x) = T$, $\gamma(v) = V$ and $\gamma(c) = [d]E$
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Name assignations: partial maps s.t. \( \gamma(x) = T, \gamma(v) = V \) and \( \gamma(c) = [d]E \)

We write \( \phi^{\text{in}} \vdash \gamma : \phi^{\text{out}} \) when \( \forall a \in \phi^{\text{out}} : \text{supp}(\gamma(a)) \subseteq \phi^{\text{in}} \).
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Name assignations: partial maps s.t. \( \gamma(x) = T, \gamma(v) = V \) and \( \gamma(c) = [d]E \)

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**Definition (syntactic category \( C_{\text{syn}} \))**

- objects are sets of names \( \phi \),
Syntactic Category

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Definition (syntactic category \( C_{\text{syn}} \))

- objects are sets of names \( \phi \),
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- composition of \( \phi^{\text{in}} \vdash \gamma : \phi \) and \( \phi \vdash \delta : \phi^{\text{out}} \): \( \gamma \circ \delta \) is the partial map from \( a \in \phi^{\text{out}} \) to \( \delta(a)\{\gamma\} \). We then have \( \phi^{\text{in}} \vdash \gamma \circ \delta : \phi^{\text{out}} \).
Syntactic Category

Names: $a = x \mid v \mid c$
Name assignations: partial maps s.t. $\gamma(x) = T, \gamma(v) = V$ and $\gamma(c) = [d]E$

We write $\phi^{in} \vdash \gamma : \phi^{out}$ when $\forall a \in \phi^{out}: \text{supp}(\gamma(a)) \subseteq \phi^{in}$.

Definition (syntactic category $C_{syn}$)

- objects are sets of names $\phi$,
- morphisms are name assignations,
- composition of $\phi^{in} \vdash \gamma : \phi$ and $\phi \vdash \delta : \phi^{out}$: $\gamma \circ \delta$ is the partial map from $a \in \phi^{out}$ to $\delta(a)\{\gamma\}$. We then have $\phi^{in} \vdash \gamma \circ \delta : \phi^{out}$,
- identity of $\{a_1, \ldots, a_k\}$ is the map $[a_1 \mapsto a_1] \cdots [a_k \mapsto a_k]$. 

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The OGS LTS ($\mathcal{L}_{\text{OGS}}$)

**Definition (Configurations)**

$G; H \in \text{CONF}$ are either passive of the shape $\langle \gamma \rangle$, or active of the shape $\langle N \mid \gamma \rangle$ with

- $N$ a named term ($[c]T$ with $c$ a continuation name);
- $\gamma$ a name assignation.
The OGS LTS ($\mathcal{L}_{\text{OGS}}$)

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$G; H \in \text{Conf}$ are either passive of the shape $\langle \gamma \rangle$, or active of the shape $\langle N | \gamma \rangle$ with

- $N$ a named term ([c]T with $c$ a continuation name);
- $\gamma$ a name assignment.

\[
\text{decomp}(NF) \text{ transform normal forms into a pair } (m, \gamma):
\]

\[
\text{decomp}(K[x]) \triangleq \{(x(c), [c \mapsto K]) | c \in \text{CNames}\}
\]

\[
\text{recomp}(m, \gamma) \text{ apply the substitution from } \gamma \text{ to get a named term:}
\]

\[
\text{recomp}(x(c), \gamma) \triangleq [c]\gamma(x)
\]
The OGS LTS ($\mathcal{L}_{\text{OGS}}$)

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**$\text{decomp}(\text{NF})$ transform normal forms into a pair $(m, \gamma)$:**

- $\text{decomp}(K[x]) \triangleq \{(x(c), [c \mapsto K]) | c \in \text{CNames}\}$
- $\text{recomp}(m, \gamma)$ apply the substitution from $\gamma$ to get a named term:
  - $\text{recomp}(x(c), \gamma) \triangleq [c]\gamma(x)$

```
\[
\begin{array}{c}
T \rightarrow U \\
\left[\begin{array}{c}
[c]T | \gamma \\
\end{array}\right] \xrightarrow{\text{eval}_{\text{ogs}}} \left[\begin{array}{c}
[c]U | \gamma \\
\end{array}\right] \\
\end{array}
\]
```

$\langle N | \gamma \rangle \xrightarrow{m}_{\text{ogs}} \langle \delta \cdot \gamma \rangle$

$\text{recomp}(m, \gamma) = N$

Towards Categorical Structures for Operational Game Semantics
We define the morphism \( \circ \) between \( \mathcal{LOGS} \) and \( \mathcal{G} \) as:

\[
\langle \gamma \rangle \circ \langle \phi^{\text{out}} \mid \phi^{\text{in}} \rangle \triangleq \phi^{\text{in}} \vdash \gamma : \phi^{\text{out}}
\]

\[
\langle N \mid \gamma \rangle \circ \langle \phi^{\text{out}} \mid \phi^{\text{in}} \rangle \triangleq \phi^{\text{in}} \vdash \gamma : \phi^{\text{out}} \land \phi^{\text{in}} \vdash N
\]

Use \( \text{CONF}[P] \) for the set of configurations \( \mathcal{G} \) satisfying \( \mathcal{G} \circ P \).
We define the morphism \( \circ \) between \( \mathcal{L}_{OGS} \) and \( \mathcal{G} \) as:

\[
\langle \gamma \rangle \circ \langle \phi^\text{out} | \phi^\text{in} \rangle \ominus \triangleq \phi^\text{in} \vdash \gamma : \phi^\text{out} \\
\langle N | \gamma \rangle \circ \langle \phi^\text{out} | \phi^\text{in} \rangle \oplus \triangleq \phi^\text{in} \vdash \gamma : \phi^\text{out} \wedge \phi^\text{in} \vdash N
\]

Use \( \text{CONF}[\mathcal{P}] \) for the set of configurations \( \mathcal{G} \) satisfying \( \mathcal{G} \circ \mathcal{P} \).

**Theorem**

The function mapping a name assignment to a strategy:

\[
\phi^\text{in} \vdash \gamma : \phi^\text{out} \rightarrow (\mathcal{L}_{OGS}, \circ, \langle \gamma \rangle)
\]

induces a functor between \( C_{syn} \) and \( C_{sem} \).
Merging in $\mathcal{L}_{\text{OGS}}$:

Let $G_P \in \text{CONF}[\langle \phi^\text{in}_P | \phi^\text{out}_P \rangle^{\kappa_P}]$ and $G_O \in \text{CONF}[\langle \phi^\text{in}_O | \phi^\text{out}_O \rangle^{\kappa_O}]$.

The merging $G_P H_P \gamma H_O \gamma_P \gamma_O G_O$ is

- $\langle (N \mid \eta) \mid \gamma \rangle$ when one of $G_P, G_O$ is active and $N$ the active term;
- $\langle \eta \mid \gamma \rangle$ when both $G_P, G_O$ are passive;

$\eta = \gamma_P \mid_{H_P} \cdot \gamma_O \mid_{H_O}$ and $\gamma = \gamma_P \mid_{\phi^\text{out}_P \setminus H_P} \cdot \gamma_O \mid_{\phi^\text{out}_O \setminus H_O}$.
Merging in $\mathcal{L}_{\text{OGS}}$:

Let $G_P \in \text{CONF}[\langle \phi_P^{\text{in}} | \phi_P^{\text{out}} \rangle^{\kappa_P}]$ and $G_O \in \text{CONF}[\langle \phi_O^{\text{in}} | \phi_O^{\text{out}} \rangle^{\kappa_O}]$. The merging $G_P H_P O \gamma H_O P G_O$ is

- $\langle (N | \eta) | \gamma \rangle$ when one of $G_P, G_O$ is active and $N$ the active term;
- $\langle \eta | \gamma \rangle$ when both $G_P, G_O$ are passive;

$\eta = \gamma_P \upharpoonright H_P O \cdot \gamma_O \upharpoonright H_O P$ and $\gamma = \gamma_P \upharpoonright \phi_P^{\text{out}} \setminus H_P O \cdot \gamma_O \upharpoonright \phi_O^{\text{out}} \setminus H_O P$.

MOGS LTS ($\mathcal{L}_{\text{MOGS}}$):

Abstract machine:

\[
T \rightarrow U
\]
\[
\langle [c]T | \gamma \rangle \rightarrow_{\text{me-op}} \langle [c]U | \gamma \rangle
\]
\[
\langle A | \gamma \rangle \xrightarrow{\text{eval}}_{\text{mogs}} \langle B | \gamma \rangle
\]
\[
\langle A | \gamma \rangle \xrightarrow{\text{sync}}_{\text{mogs}} \langle B | \gamma \rangle
\]

$\text{recomp}(m, \gamma) = N$

\[
\langle \eta | \gamma \rangle \xrightarrow{m}_{\text{mogs}} \langle (N | \eta) | \gamma \rangle
\]

$\langle \langle N | \eta \rangle | \gamma \rangle \xrightarrow{m}_{\text{mogs}} \langle \eta | \delta \cdot \gamma \rangle$

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A Functor between $\mathcal{C}_{\text{syn}}$ and $\mathcal{C}_{\text{sem}}$

**Theorem (Chain of bisimulations)**

- We have a bisimulation between $(\mathcal{L}_{\text{MOGS}}, \circledast)$ and $(\mathcal{L}_{\text{OGS}}, \circledast)$. 

Towards Categorical Structures for Operational Game Semantics 13/14
A Functor between $\mathcal{C}_{syn}$ and $\mathcal{C}_{sem}$

Theorem (Chain of bisimulations)

- We have a bisimulation between $(\mathcal{L}_{MOGS}, \circ)$ and $(\mathcal{L}_{OGS}, \circ)$.
- The function mapping $G_P \parallel H_O \gamma H_O G_O$ in $G_P H_O \gamma H_O G_O$ is a bisimulation between the parallel composition of two copies of $(\mathcal{L}_{OGS}, \circ)$ and $(\mathcal{L}_{MOGS}, \circ)$. 
Theorem (Chain of bisimulations)

- We have a bisimulation between \((\mathcal{L}_{\text{MOGS}}, \circ)\) and \((\mathcal{L}_{\text{OGS}}, \circ)\).

- The function mapping \(G_P H_P \parallel H_O P G_O\) in \(G_P H_P \triangleright H_O P G_O\) is a bisimulation between the parallel composition of two copies of \((\mathcal{L}_{\text{OGS}}, \circ)\) and \((\mathcal{L}_{\text{MOGS}}, \circ)\).

- We have a bisimulation between the parallel composition of two copies of \((\mathcal{L}_{\text{OGS}}, \circ)\) and \((\mathcal{L}_{\text{OGS}}, \circ)\) itself.
A Functor between $\mathcal{C}_{\text{syn}}$ and $\mathcal{C}_{\text{sem}}$

**Theorem (Chain of bisimulations)**

- We have a bisimulation between $(\mathcal{L}_{\text{MOGS}}, \circ)$ and $(\mathcal{L}_{\text{OGS}}, \circ)$.
- The function mapping $G_P H_P \parallel \text{HOP } G_O$ in $G_P H_P \gamma \text{HOP } G_O$ is a bisimulation between the parallel composition of two copies of $(\mathcal{L}_{\text{OGS}}, \circ)$ and $(\mathcal{L}_{\text{MOGS}}, \circ)$.
- We have a bisimulation between the parallel composition of two copies of $(\mathcal{L}_{\text{OGS}}, \circ)$ and $(\mathcal{L}_{\text{OGS}}, \circ)$ itself.

**Theorem**

The function mapping a name assignment to a strategy:

$$\phi^{\text{in}} \vdash \gamma : \phi^{\text{out}} \rightarrow (\mathcal{L}_{\text{OGS}}, \circ, \langle \gamma \rangle)$$

induces a functor between $\mathcal{C}_{\text{syn}}$ and $\mathcal{C}_{\text{sem}}$. 
Conclusion

What have we done?

- Game LTS
- Parallel composition
- Semantic category (based on strategies)
- Syntactic category (based on name assignations)
- OGS LTS using name assignations
- a morphism between OGS LTS and game LTS
- a functor between the two categories
Conclusion

What have we done?

- Game LTS
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- OGS LTS using name assignations
- a morphism between OGS LTS and game LTS
- a functor between the two categories

What we will do?

- A denotational model
- A call-by-need version