Indexed differential linear logic and Laplace transform

LHC Days 2024
June 5, 2024

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Outline

Background:

Our contributions:
Outline

Background:

- **Linear logic via its semantics**
- **Differential** linear logic
- Its extension to **D-DiLL**
- **Graded** linear logic

Our contributions:
Outline

Background:

- Linear logic via its semantics
- Differential linear logic
- Its extension to D-DiLL
- Graded linear logic

Our contributions:

- A finitary differential linear logic, graded with differential operators
- A semiring of differential operators
- Laplace transform for operators
1. Background
The smooth semantics

Formulas:

- Each MALL formula is a finite dimensional vector space:
  \[ [1] := \mathbb{R} \quad [A \otimes B] := [A] \otimes [B] \quad [A \oplus B] := [A] \cup [B] \quad \ldots \]

- Exponentials are interpreted by infinite dimensional vector spaces:
  - \([?A] := \mathcal{C}^{\infty}([A]', \mathbb{R}) \) (functions)
  - \([!A] := \mathcal{C}^{\infty}([A], \mathbb{R})' \) (distributions)

- Negation is duality: \([A^\perp] := [A]' = \mathcal{L}([A], \mathbb{R})\)
The smooth semantics

Formulas:

- Each MALL formula is a finite dimensional vector space:
  \[ [1] := \mathbb{R} \quad [A \otimes B] := [A] \otimes [B] \quad [A \oplus B] := [A] \oplus [B] \quad \ldots \]

- Exponentials are interpreted by infinite dimensional vector spaces:
  - \[ [?A] := C^\infty([A]', \mathbb{R}) \quad (functions) \]
  - \[ ![A] := C^\infty([A], \mathbb{R})' \quad (distributions) \]

- Negation is duality: \[ [A^\perp] := [A]' = \mathcal{L}([A], \mathbb{R}) \]

Proofs:

- Each proof is a linear map between the interpretation of the formulas.
  - \[ A \Rightarrow B = ![A] \rightarrow [B] \] is \[ C^\infty(A, B) \cong \mathcal{L}(![A], B) \]

- The dereliction states that \[ \mathcal{L}(A, B) \subseteq C^\infty(A, B) \] : it forgets the linearity.
Differential Linear Logic


Differential linear logic is about linear extraction of a proof

\[
\ell : A \vdash B \\
\ell : !A \vdash B \\
D_0(f) : A \vdash B \quad \bar{d}
\]

\[
f \in C^\infty(\mathbb{R}, \mathbb{R})
\]

\[
d(f)(0)
\]
Differential Linear Logic

- Other rules has to be added (cut-elimination)

\[
\begin{align*}
\Gamma & \vdash \Gamma, \vdash \Gamma, ?A, ?A \\
\Gamma, ?A & \vdash \Gamma, A \\
\Gamma & \vdash \Gamma, ?A \\
?\Gamma & \vdash ?\Gamma, !A
\end{align*}
\]
Other rules has to be added (cut-elimination)

\[
\begin{align*}
\vdash \Gamma & \quad \vdash \Gamma, ?A, ?A & \quad \vdash \Gamma, A & \quad \vdash ?\Gamma, !A \\
\vdash \Gamma, ?A & \quad \vdash \Gamma, ?A & \quad \vdash \Gamma, ?A & \quad \vdash ?\Gamma, !A \\
\vdash !A & \quad \vdash !A & \quad \vdash \Delta, !A & \quad \vdash !A
\end{align*}
\]
Differential Linear Logic

- Other rules has to be added (cut-elimination)

\[
\begin{align*}
\Gamma & \vdash \Gamma, \text{cst}_1 : ?A \\
\Gamma & \vdash \Gamma, f : ?A, g : ?A \\
\Gamma & \vdash \Gamma, \psi : !A, \Delta, \phi : !A \\
\Gamma & \vdash \Gamma, D_0(\_)(v) : !A
\end{align*}
\]
Differential Linear Logic

- Other rules has to be added (cut-elimination)

\[
\begin{align*}
\Gamma \vdash & \quad \Gamma, \text{cst}_1 : ?A \\
\Gamma, f : ?A, g : ?A \vdash & \quad \Gamma, f \cdot g : ?A \\
\Gamma \vdash & \quad \Gamma, x : A \\
? \Gamma, x : A \vdash & \quad ? \Gamma, \delta_x : !A \\
\Gamma, \psi : !A \vdash & \quad ? \Gamma, \psi \ast \phi : !A \\
\Gamma, \Delta, \psi \ast \phi : !A \vdash & \quad \Gamma, D_0(\_)(v) : !A
\end{align*}
\]

- They have nice mathematical interpretation (differential calculus)

\(\bar{d}/\bar{p}\) is the chain rule

\(\ldots\)
From $D_0$ to differential equations

<table>
<thead>
<tr>
<th>Linearity &amp; Forgets</th>
<th>Linearity &amp; Applies</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ell : A \vdash B$</td>
<td>$f : !A \vdash B$</td>
</tr>
<tr>
<td>$\ell : !A \vdash B$</td>
<td>$D_0(f) : A \vdash B$</td>
</tr>
</tbody>
</table>

Forgets linearity

Applies $D_0$
From $D_0$ to differential equations

\[
\ell : A \vdash B \\
\ell : !A \vdash B \\
\text{d} \\
\text{Forgets linearity}
\]

\[
f : !A \vdash B \\
D_0(f) : A \vdash B \\
\text{d} \\
\text{Applies } D_0
\]

Solution of $D_0(\_ ) = \ell$ ?
That is $\ell$ since $D_0(\ell) = \ell$
From $D_0$ to differential equations

\[
\ell : A \vdash B \\
\ell : !A \vdash B \\
\text{Forgets linearity}
\]

\[
f : !A \vdash B \\
D_0(f) : A \vdash B \\
\text{Applies } D_0
\]

Solution of $D_0(\_ \_ ) = \ell$ ?
That is $\ell$ since $D_0(\ell) = \ell$

\[
\ell : A \vdash B \\
f : !A \vdash B \\
\text{Forgets linearity} \\
\text{Solves } D
\]

\[
f : !A \vdash B \\
D(f) : A \vdash B \\
\text{Apply } D_0 \\
\text{Applies } D
\]
From $D_0$ to differential equations

\[
\frac{\ell : A \vdash B}{\ell : !A \vdash B} \quad \text{d}
\]

Forgets linearity

\[
\frac{f : !A \vdash B}{D_0(f) : A \vdash B} \quad \text{d}
\]

Applies $D_0$

Solution of $D_0(\_ \_ ) = \ell$ ?
That is $\ell$ since $D_0(\ell ) = \ell$

\[
\frac{\ell : A \vdash B}{f : !A \vdash B} \quad \text{d}
\]

Solves $D$

\[
\frac{f : !A \vdash B}{D(f) : A \vdash B} \quad \text{d}
\]

Apply $D_0$
Applies $D$

Solution? (LPDO)
From $D_0$ to differential equations

\[
\ell : A \vdash B \\
\ell : !A \vdash B \\
\text{Forgets linearity}
\]

\[
f : !A \vdash B \\
D_0(f) : A \vdash B \\
\text{Applies $D_0$}
\]

Solution of $D_0(\_ \_)=\ell$ ?
That is $\ell$ since $D_0(\ell)=\ell$

\[
\ell : A \vdash B \\
f : !A \vdash B \\
\text{Forgets linearity}
\]

\[
f : !A \vdash B \\
D(f) : A \vdash B \\
\text{Apply $D_0$}
\]

Solution? (LPDO)  Type?
A **LPDO** is a linear operator defined as

\[ D = \sum_{\alpha \in \mathbb{N}^n} a_{\alpha} \frac{\partial |\alpha|}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} \quad (a_{\alpha} \in \mathbb{R}) \]

- A LPDO acts on smooth maps, or distributions.
- A **fundamental solution** is a distribution \( \Phi_D \) s.t. \( D(\Phi_D) = \delta_0 \)

Examples of LPDOs: \( D : f \mapsto \frac{\partial}{\partial x_1} f + 3 \frac{\partial^2}{\partial x_1 \partial x_3} f \), or the heat equation.

**Theorem (Malgrange-Ehrenpreis, 50’s)**

Each LPDO \( D \) has a unique fundamental solution \( \Phi_D \).
A logical account for LPDE. Kerjean (2018)

- Two new exponentials connectives: $!_{\mathcal{D}}A$ and $?_{\mathcal{D}}A$
- Their interpretations in the smooth semantics:
  - $[?_{\mathcal{D}}A] := D(C^\infty([A]', \mathbb{R}))$ (parameters)
  - $[!_{\mathcal{D}}A] := (D(C^\infty([A], \mathbb{R})))'$ (solutions)
- We have $!_{\mathcal{D}_0}A \simeq A$

The exponential rules of D-DiLL

\[
\begin{align*}
\frac{\Gamma}{\Gamma, ?_{\mathcal{D}}A} & \quad w_D & \frac{\Gamma, ?A, ?_{\mathcal{D}}A}{\Gamma} & \quad c_D \\
\frac{\Gamma, ?_{\mathcal{D}}A}{\Gamma, ?A} & \quad d_D & \frac{\Gamma, !A}{\Delta, !_{\mathcal{D}}A} & \quad \bar{c}_D \\
\frac{!_{\mathcal{D}}A}{w_D} & & \frac{\Gamma, !_{\mathcal{D}}A}{\Gamma, \Delta, !_{\mathcal{D}}A} & \quad \bar{d}_D
\end{align*}
\]
Did we solve our issues?

\[
\ell : A \vdash B \quad \text{solution of } D : !A \vdash B \\
Solves \ D
\]

\[
f : !A \vdash B \quad \text{d}
\]

\[
D(f) : A \vdash B \\
\text{Applies } D
\]
DiLL indexed by a LPDOcc

Did we solve our issues?

\[
\frac{f : !_D A \vdash B}{f \ast \Phi_D : !A \vdash B} \quad \text{Solves } D
\]

\[
\frac{f : !A \vdash B}{D(f) : A \vdash B} \quad \text{Applies } D
\]

\[d\]
DiLL indexed by a LPDOcc

Did we solve our issues?

\[
\begin{align*}
  f : !_D A \vdash B & \quad \overset{d}{\Rightarrow} \\
  f \ast \Phi_D : ! A \vdash B & \quad \text{Solves } D
\end{align*}
\]

\[
\begin{align*}
  f : ! A \vdash B & \quad \overset{d}{\Rightarrow} \\
  D(f) : !_D A \vdash B & \quad \text{Applies } D
\end{align*}
\]
DiLL indexed by a LPDOcc

Did we solve our issues?

\[
\begin{align*}
\frac{f : !D A \vdash B}{f \cdot \Phi_D : !A \vdash B} & \quad \text{Solves } D \\
\frac{f : !A \vdash B}{D(f) : !D A \vdash B} & \quad \text{Applies } D
\end{align*}
\]

A graded version?

- Our exponential is indexed, can we connect with other frameworks?
- Is there an interaction?
- LPDOcc are well-behaved: \( \Phi_{D_1 \circ D_2} = \Phi_{D_1} * \Phi_{D_2} \)
Graded linear logic

- A core quantitative coeffect calculus. Brunel et. al (2014)

Exponential rules of $B_{SLL}$

\[
\begin{align*}
\Gamma \vdash B & \quad \Gamma, !A, !A \vdash B \\
\Gamma, !A & \vdash B \\
\Gamma, !x!y A & \vdash B \\
\Gamma, A & \vdash B \\
\Gamma, !1!A & \vdash B \\
\Gamma, !x!y A & \vdash B \\
\Gamma, !x!y A & \vdash B \\
\Gamma, !x!y A & \vdash B \\
\end{align*}
\]
Graded linear logic

- **A core quantitative coeffect calculus.** Brunel et. al (2014)

- **Combining Effects and Coeffects via Grading.** Gaboardi et. al (2016)

### Exponential rules of $B_{\text{SLL}}$

<table>
<thead>
<tr>
<th>Rule</th>
<th>Context</th>
<th>Premises</th>
</tr>
</thead>
<tbody>
<tr>
<td>w</td>
<td>$\Gamma \vdash B$</td>
<td>$\Gamma, !_0 A \vdash B$</td>
</tr>
<tr>
<td></td>
<td>$\Gamma, !_0 A \vdash B$</td>
<td>$\Gamma, !x A, !y A \vdash B$</td>
</tr>
<tr>
<td>c</td>
<td>$\Gamma, A \vdash B$</td>
<td>$\Gamma, !x+y A \vdash B$</td>
</tr>
<tr>
<td></td>
<td>$\Gamma, !x+y A \vdash B$</td>
<td>$\Gamma, !1 A \vdash B$</td>
</tr>
<tr>
<td>p</td>
<td>$\Gamma, !x A \vdash B$</td>
<td>$\Gamma, !x_1 A_1, \ldots, !x_n A_n \vdash B$</td>
</tr>
<tr>
<td></td>
<td>$\Gamma, !x_1 A_1, \ldots, !x_n A_n \vdash B$</td>
<td>$\Gamma, !x_1 \times y A_1, \ldots, !x_n \times y A_n \vdash !y B$</td>
</tr>
<tr>
<td>d</td>
<td>$\Gamma, A \vdash B$</td>
<td>$\Gamma, !x_1 \times y A_1, \ldots, !x_n \times y A_n \vdash !y B$</td>
</tr>
<tr>
<td>d_I</td>
<td>$\Gamma, A \vdash B$</td>
<td>$\Gamma, !1 A \vdash B$</td>
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<td></td>
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Additive law
Graded linear logic

A core quantitative coeffect calculus. Brunel et. al (2014)


Exponential rules of $B_{SLL}$

\[
\frac{\Gamma \vdash B}{\Gamma, !_0 A \vdash B} \quad w \quad \frac{\Gamma, !x A, !y A \vdash B}{\Gamma, !x+y A \vdash B} \quad c \quad \frac{\Gamma, A \vdash B}{\Gamma, !_1 A \vdash B} \quad d
\]

\[
\frac{!x_1 A_1, \ldots, !x_n A_n \vdash B}{!x_1 \times y A_1, \ldots, !x_n \times y A_n \vdash !_y B} \quad p \quad \frac{\Gamma, !x A \vdash B \quad x \leq y}{\Gamma, !_y A \vdash B} \quad d_I
\]

Multiplicative law  Additive law
Graded linear logic

A core quantitative coeffect calculus. Brunel et. al (2014)


Exponential rules of $B_{SLL}$

$$
\frac{\Gamma \vdash B}{\Gamma, !0 A \vdash B} \quad \frac{\Gamma, !x A, !y A \vdash B}{\Gamma, !x + y A \vdash B} \quad \frac{\Gamma, A \vdash B}{\Gamma, !1 A \vdash B}
$$

$$
\frac{!x_1 A_1, \ldots, !x_n A_n \vdash B}{!x_1 \times y A_1, \ldots, !x_n \times y A_n \vdash !y B} \quad \frac{\Gamma, !x A \vdash B \quad x \leq y}{\Gamma, !y A \vdash B}
$$

- Multiplicative law
- Additive law
- Order

- $w$
- $c$
- $d$
- $d_I$
Graded linear logic

A core quantitative coeffect calculus. Brunel et. al (2014)


Exponential rules of $B_{SLL}$

Multiplicative law

Additive law

Order

$(\mathcal{S}, +, 0, \times, 1, \leq)$ is an ordered semiring
Graded linear logic

- A core quantitative coeffect calculus. Brunel et. al (2014)

**Exponential rules of BSLL**

\[
\begin{align*}
\frac{\Gamma \vdash B}{\Gamma, !0A \vdash B} & \quad w \\
\frac{\Gamma, !xA, !yA \vdash B}{\Gamma, !x+yA \vdash B} & \quad c \\
\frac{!x_1A_1, \ldots, !x_nA_n \vdash B}{!x_1 \times yA_1, \ldots, !x_n \times yA_n \vdash !yB} & \quad p \\
\frac{\Gamma, !_1A \vdash B}{\Gamma, !A \vdash B} & \quad d \\
\frac{\Gamma, !xA \vdash B}{\Gamma, !yA \vdash B} & \quad d_I \\
\end{align*}
\]

- Multiplicative law
- Additive law
- Order

\((\mathcal{S}, +, 0, \times, 1, \leq)\) is an ordered semiring

- Type system for ressource consumption
- Coeffect analysis
2. A graded differential linear logic
The logic $\text{DB}_{\mathcal{S}LL}$

- A syntactical differentiation of $\mathcal{B}_{\mathcal{S}LL}$

**The exponential rules of $\text{DB}_{\mathcal{S}LL}$**

\[
\begin{align*}
\vdash & \Gamma \quad \vdash & \Gamma, \, ?_0 A & w \\
\vdash & \Gamma \quad \vdash & \Gamma, \, ? x A, \, ? y A & c \\
\vdash & \Gamma \quad \vdash & \Gamma, \, ? x/y A & d_I \\
\vdash & \Gamma \quad \vdash & \Gamma, A & d \\
\end{align*}
\]

When $S$ is additive splitting, and the order is define through the sum, $\text{DB}_{\mathcal{S}LL}$ enjoys a cut elimination procedure.
Let $\mathcal{D}$ be the set of LPDOcc.

$$\mathcal{D} \cong \mathbb{R}[X_1, \ldots, X_n, \ldots]$$

$$\left( D = \sum_{\alpha \in \mathbb{N}} k_\alpha \frac{\partial |\alpha|}{\partial x_1^{\alpha_1} \ldots x_n^{\alpha_n}} \right) \mapsto \left( P = \sum_{\alpha \in \mathbb{N}} k_\alpha X_1^{\alpha_1} \ldots X_n^{\alpha_n} \right)$$

**Proposition**

The set of LPDOcc, endowed with composition, is an additive splitting commutative monoid.
An indexed differential linear logic

**Exponential rules of IDiLL**

\[
\begin{align*}
&\vdash \Gamma \quad w_I \\
&\vdash \Gamma, ?_{D1}A, ?_{D2}A \\
&\vdash \Gamma, ?_{D1 \circ D2}A \\
&\vdash \Gamma, ?_{D1}A \\
&\vdash \Gamma, ?_{D1 \circ D2}A \\
&\vdash \Gamma \quad d_I \\
&\vdash \Gamma, !_{D1}A \\
&\vdash \Delta, !_{D2}A \\
&\vdash \Gamma, \Delta, !_{D1 \circ D2}A \\
&\vdash \Gamma, !_{D1}A \\
&\vdash \Gamma, !_{D1 \circ D2}A \\
&\vdash \Gamma \quad \bar{d}_I \\
&\vdash \Gamma, \bar{D}1A \\
&\vdash \Gamma, \bar{D}1 \circ \bar{D}2A \\
&\vdash \Gamma, \bar{D}1A \\
&\vdash \Gamma, \bar{D}1 \circ \bar{D}2A \\
&\vdash \Gamma \quad \bar{w}_I \\
&\vdash \Gamma, \bar{D}1A \\
&\vdash \Gamma, \bar{D}1 \circ \bar{D}2A
\end{align*}
\]

From \(d_D\) to \(d_I\): syntax has to change, and semantics as well

- \([!_{D}A] = D(C^\infty([A], \mathbb{R})') \quad (solutions) \quad (parameters)\)
- \([?_{D}A] = D^{-1}(C^\infty([A]', \mathbb{R})) \quad (parameters) \quad (solutions)\)

The duality transforms solutions into parameters
The smooth semantics for IDiLL

**Definition**

\[ w : \begin{cases} \mathbb{R} & \to \ ?_{id}E \\ 1 & \mapsto \text{cst}_1 \end{cases} \quad \bar{w} : \begin{cases} \mathbb{R} & \to !_{id}E \\ 1 & \mapsto \delta_0 \end{cases} \]

\[ c : \begin{cases} ?_{D_1}E \otimes ?_{D_2}E & \to ?_{D_1 \circ D_2}E \\ f \otimes g & \mapsto \Phi_{D_1 \circ D_2} \ast (D_1(f) \cdot D_2(g)) \end{cases} \]

\[ \bar{c} : \begin{cases} !_{D_1}E \otimes !_{D_2}E & \to !_{D_1 \circ D_2}E \\ \psi \otimes \phi & \mapsto \psi \ast \phi \end{cases} \]

\[ d_I : \begin{cases} ?_{D_1}E & \to ?_{D_1 \circ D_2}E \\ f & \mapsto \Phi_{D_2} \ast f \end{cases} \quad \bar{d}_I : \begin{cases} !_{D_1}E & \to !_{D_1 \circ D_2}E \\ \psi & \mapsto \psi \circ D_2 \end{cases} \]

**Theorem**

The smooth semantics is **compatible** with the cut-elimination procedure.
3. Laplace transform and promotion rule
The (graded) promotion rule

- In linear logic

\[
\begin{align*}
!A_1, \ldots, !A_n \vdash B & \quad \frac{!A \xrightarrow{f} B}{!A \xrightarrow{pA} !!A} \\
\vdash !B & \quad \frac{!A \xrightarrow{!f} !B}{!A \xrightarrow{pA} !!A} \\
\end{align*}
\]
The (graded) promotion rule

- In linear logic

\[
\begin{array}{c}
!A_1, \ldots, !A_n \vdash B \\
!A_1, \ldots, !A_n \vdash !B \\
\end{array}
\]

\[
\begin{array}{c}
!A \xrightarrow{f} B \\
!A \xrightarrow{p^A} !!A \\
!!A \xrightarrow{!f} !B \\
\end{array}
\]

For \( f \in C(A, B) \), \( g \in C(B, C) \), \( f; g \) is:

\[
\begin{array}{c}
!A \xrightarrow{p^A} !!A \\
!!A \xrightarrow{!f} !B \\
!B \xrightarrow{g} C \\
\end{array}
\]
The (graded) promotion rule

- In linear logic

\[
\begin{align*}
!A_1, \ldots, !A_n &\vdash B \\
\therefore !A_1, \ldots, !A_n &\vdash !B \\
\end{align*}
\]

By \text{p}

\[
\begin{align*}
!A &\xrightarrow{f} B \\
\therefore !A &\xrightarrow{p_A} !!A \\
\therefore !f &\xrightarrow{} !B
\end{align*}
\]

For \( f \in C(A, B), g \in C(B, C), \ f; g \) is:

\[
\begin{align*}
!A &\xrightarrow{p_A} !!A \\
\therefore !f &\xrightarrow{} !B \\
g &\xrightarrow{} C
\end{align*}
\]

- In graded linear logic

\[
\begin{align*}
!y_1 A_1, \ldots, !y_n A_n &\vdash B \\
\therefore !x \times y_1 A_1, \ldots, !x \times y_n A_n &\vdash !x !B \\
\end{align*}
\]

By \text{p}

\[
\begin{align*}
!y A &\xrightarrow{f} B \\
\therefore !x \times y A &\xrightarrow{p_A, x, y} !x !y A \\
\therefore !xf &\xrightarrow{} !x !B
\end{align*}
\]
Can we define a promotion?

Three questions:

1. Can we compose solutions?
2. Can we multiply differential operators?
3. Can we interpret higher-order?
Can we define a promotion?

Three questions:

1. Can we compose solutions?
2. Can we multiply differential operators?
3. Can we interpret higher-order?

- The function $x \mapsto e^x$ is a solution of $f' - f = 0$.
- But $e^{e^x}$ is not a solution of $D(f) = 0$ (even with polynomial coeffs!)
- Not really a problem for us: each map can be chosen as a parameter.
- If $f$ solution of $D_1$ and $g$ solution of $D_2$, $g \circ f$ solution of ?
Can we define a promotion?

Three questions:

1. Can we compose solutions?
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3. Can we interpret higher-order?
Can we define a promotion?

Three questions:

1. Can we compose solutions?
2. Can we multiply differential operators?
3. Can we interpret higher-order?

- Our sum is the composition of operators
- Can we define $\odot$ such that $(\mathcal{D}, \circ, \odot)$ is a semiring?

\[
\frac{\vdash \Gamma, B \bot}{\vdash \Gamma, ?_1 B \bot} \quad \frac{\vdash ?_x A, B}{\vdash ?_1 \times x A, !_1 B} \quad \text{p} \quad \quad \frac{\vdash \Gamma, B \bot}{\vdash \Gamma, ? \times_x A} \quad \text{cut}
\]

\[
\frac{\vdash \Gamma, ? \times_x A}{\vdash \Gamma, ?_1 \times_x A} \quad \text{cut}
\]
Can we define a promotion?

Three questions:

1. Can we compose solutions?
2. Can we multiply differential operators?
3. Can we interpret higher-order?

**Definition**

A differential semiring is a tuple \((S, 0, 1, +, \times)\) s.t.

- \(0 \times x = 0\) \((w)\)
- \((x + y)z = xz + yz\) \((c)\)
- \(1 \times x = x\) \((d)\)
- \(x \leq y \Rightarrow xz \leq yz\) \((d_I)\)
- \(x(yz) = (xy)z\) \((p)\)

- \(xy = 0 \Rightarrow x = 0 \text{ or } y = 0\) \((\bar{w})\)
- \(\text{mult. split. } (x_1 x_2 = y_1 + y_2)\) \((\bar{c})\)
- \(? ? ? \) (chain rule) \((\bar{d})\)
- \(? ? ? \) (graded chain rule) \((\bar{d}_I)\)

The axiom \((d_I)\) is implied by \((c)\), thanks to the definition of the order.
Can we define a promotion?

Three questions:

1. Can we compose solutions?
2. Can we multiply differential operators?
3. Can we interpret higher-order?

We define $\odot$ as:

$$
\left( \sum_{\alpha \in \mathbb{N}^n} a_{\alpha} \partial^{\alpha} \right) \odot \left( \sum_{\beta \in \mathbb{N}^n} b_{\beta} \partial^{\beta} \right) = \sum_{\alpha, \beta \in \mathbb{N}^n} a_{\alpha}(b_{\beta}) |^{\alpha} \partial^{\alpha \beta}
$$

This verifies $(w), (c), (d_I)$ and $(d)$ or $(p)$.
What about costructural axioms?
Can we define a promotion?

Three questions:

1. Can we compose solutions?
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Can we define a promotion?

Three questions:

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An overview of models of DiLL

<table>
<thead>
<tr>
<th>Model</th>
<th>Reflexivity</th>
<th>Smoothness</th>
<th>Higher-order</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kothe spaces</td>
<td>✓</td>
<td>✗</td>
<td>✓</td>
</tr>
<tr>
<td>Convenient spaces</td>
<td>✗</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Nuclear Frechet spaces</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
</table>
Can we define a promotion?

Three questions:

1. Can we compose solutions?
2. Can we multiply differential operators?
3. Can we interpret higher-order?

We need linearly independent families to interpret partial derivatives:

- For finitary formulas, we are isomorphic to $\mathbb{R}^n$.
- For $(E, V)$, we define $!_D (E, V)$ as $(!_DE, !_DV)$, where

$$V = (x_1, \ldots, x_n, \ldots) \rightarrow !_DV = (\delta_{x_1}, \ldots, \delta_{x_n}, \ldots)$$

- For MALL connectives over exponential formulas, usual constructions work.
The Laplace transform for distributions:

\[ \mathcal{L} : \begin{cases} !A & \rightarrow \ ?A' \\ \psi & \mapsto (\ell \in A' \mapsto \psi(x \in A \mapsto e^{\ell(x)})) \end{cases} \]

\[ \mathcal{L}(\delta_0) = \text{cst}_1 \]
\[ \mathcal{L}(\psi * \phi) = \mathcal{L}(\psi).\mathcal{L}(\phi) \]
\[ \mathcal{L}(D_0(\_)(v)) = \text{eval}_v \]

*Laplace transform turns costructural rules into structural ones.*
Laplace transform and differential operators

For $P = \sum_{\alpha} a_\alpha X^\alpha$, we note $P(\partial) = \sum_{\alpha} a_\alpha \partial^\alpha$.

\begin{align*}
\mathcal{L}(P(\partial)(\psi)) &= P.\mathcal{L}(\psi) \\
\mathcal{L}(P.\psi) &= P(\partial)(\mathcal{L}(\psi))
\end{align*}
Laplace transform and differential operators

For $P = \sum_\alpha a_\alpha X_\alpha$, we note $P(\partial) = \sum_\alpha a_\alpha \partial^\alpha$.

$L(P(\partial)(\psi)) = P.L(\psi)$  \hspace{1cm}  $L(P.\psi) = P(\partial)(L(\psi))$

**PARAMETERS**

$w, c, d, p$

**SOLUTIONS**

$\tilde{w}, \tilde{c}, \tilde{d}$

18
A new syntax

\[
\frac{\vdash \Gamma}{\vdash \Gamma, ?_PA} \quad w
\]

\[
\frac{\vdash \Gamma, ?_PA, ?QA}{\vdash \Gamma, ?_PQA} \quad c
\]

\[
\frac{\vdash \Gamma, ?_PA}{\vdash \Gamma, ?_PQA} \quad d_I
\]

\[
\frac{\vdash \Gamma}{\vdash !_P(\partial)A} \quad \bar{w}
\]

\[
\frac{\vdash \Gamma, !_P(\partial)A}{\vdash \Delta, !_Q(\partial)A} \quad \bar{c}
\]

\[
\frac{\vdash \Delta, !_Q(\partial)A}{\vdash \Gamma, \Delta, !_PQA} \quad \bar{d}_I
\]

\[
\frac{\vdash ?_P(\partial)A, B}{\vdash ?(Q \odot P)(\partial)A, !_Q(\partial)B} \quad p
\]
A new semantics

\[ w : \begin{cases} \mathbb{R} & \rightarrow ?_P A \\
1 & \mapsto P.\text{cst}_1 \end{cases} \]

\[ c : \begin{cases} ?_P A \otimes ?_Q A & \rightarrow ?_{PQ} A \\
f \otimes g & \mapsto f \cdot g \end{cases} \]

\[ d_I : \begin{cases} ?_P A & \rightarrow ?_{PQ} A \\
f & \mapsto Qf \end{cases} \]

- The contraction rule is the one from DiLL
- We do not need fundamental solutions (other LPDO?)

\[ \bar{w} : \begin{cases} \mathbb{R} & \rightarrow !_P(\partial) A \\
1 & \mapsto P(\partial) \circ \delta_0 \end{cases} \]

\[ \bar{c} : \begin{cases} !_P(\partial) A \otimes !_Q(\partial) A & \rightarrow !_{(PQ)(\partial)} A \\
\psi \otimes \phi & \mapsto \psi \ast \phi \end{cases} \]

\[ \bar{d}_I : \begin{cases} !_P(\partial) A & \rightarrow !_{(PQ)(\partial)} A \\
\psi & \mapsto Q(\partial) \circ \psi \end{cases} \]
Conclusion

Take away:

- Promotion free logic, which interprets LPDOcc with solutions and parameters
- Algebraic structure for structural rules
- Laplace transform between parameters and solutions

Future works:

- A double indexed syntax for our logic
- What about the categorical semantics?
- Can we extend this to other operators? D-finite/holonomic functions?

QUESTIONS?