Finitely accessible arboreal adjunctions and Hintikka formulae

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LHC days 2024
A reformulation of an old example (1/2)


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A reformulation of an old example (1/2)

Situation

\[ \langle \overline{x} \mid \varphi \rangle \]

where

- \( \overline{x} = x_1, \ldots, x_n \)
- \( \varphi \) finite conjunction of constraints

\[ ((x_i < x_j) \text{ or } (x_i = x_j)) \]

A reformulation of an old example (1/2)

Situation

\[ \langle \overline{x} \mid \varphi \rangle \xrightarrow{m} (M, <_M) \]

where

- \( \overline{x} = x_1, \ldots, x_n \)
- \( \varphi \) finite conjunction of constraints \(((x_i < x_j) \text{ or } (x_i = x_j))\)
- \( m \) is an order embedding:
  
  \[
  m(x_i) <_M m(x_j) \iff (x_i < x_j) \text{ in } \varphi \\
  m(x_i) = m(x_j) \iff (x_i = x_j) \text{ in } \varphi
  \]


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A reformulation of an old example (2/2)

Let \((M, <_M)\) and \((N, <_N)\) be dense linear orders without end points.

(e.g. \((\mathbb{Q}, <)\) and \((\mathbb{R}, <))\)

A reformulation of an old example (2/2)

Let \((M, <_M)\) and \((N, <_N)\) be dense linear orders without end points.

(e.g. \((\mathbb{Q}, <)\) and \((\mathbb{R}, <)\))

Ehrenfeucht-Fraïssé game

(played by Spoiler and Duplicator)

\[
\langle \overline{X} \mid \varphi \rangle \quad \begin{array}{c} \leftarrow \end{array} \quad \begin{array}{c} \overline{X} \mid \varphi \rangle \quad \begin{array}{c} \rightarrow \end{array} 
\end{array}
\]

A reformulation of an old example (2/2)

Let \((M, <_M)\) and \((N, <_N)\) be dense linear orders without end points.
(e.g. \((\mathbb{Q}, <)\) and \((\mathbb{R}, <)\))

Ehrenfeucht-Fraïssé game
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(played by Spoiler and Duplicator)

A reformulation of an old example (2/2)

Let $(M, <_M)$ and $(N, <_N)$ be dense linear orders without end points.
(e.g. $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$)

Ehrenfeucht-Fraïssé game
(played by Spoiler and Duplicator)

\[ \langle x \mid \varphi \rangle \quad \text{or symmetrically.} \]

A reformulation of an old example (2/2)

Let \((M, <_M)\) and \((N, <_N)\) be dense linear orders without end points. (e.g. \((\mathbb{Q}, <)\) and \((\mathbb{R}, <))\)

Ehrenfeucht-Fraïssé game (played by Spoiler and Duplicator)

\[
\langle x \mid \varphi \rangle \quad \langle x, x' \mid \varphi \land \varphi' \rangle
\]

or symmetrically.

▶ Duplicator wins since they can always respond.

A reformulation of an old example (2/2)

Let \((M, <_M)\) and \((N, <_N)\) be dense linear orders without end points.

(e.g. \((\mathbb{Q}, <)\) and \((\mathbb{R}, <))\)

**Ehrenfeucht-Fraïssé game**

(played by Spoiler and Duplicator)

\[
\langle \overline{X} \mid \varphi \rangle \\
\langle \overline{X}, x' \mid \varphi \land \varphi' \rangle
\]

\(M\) \[\text{(Spoiler)}\] \[\langle \overline{X}, x' \mid \varphi \land \varphi' \rangle \[\text{(Duplicator)}\] \(N\)

- Duplicator wins since they can always respond.

**Corollary (\ldots, Karp (1965))**

\((M, <_M)\) and \((N, <_N)\) are equivalent in \(\mathcal{L}_\infty(<)\).

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Toward game comonads
Toward game comonads: turn plays into structures
Toward game comonads: turn plays into structures

Ehrenfeucht-Fraïssé games

Play

\[ M \rightarrow N \]

\[ \langle \rangle \]

\[ \langle x_1 \mid \varphi_1 \rangle \]

\[ \langle x_1, x_2 \mid \varphi_1 \land \varphi_2 \rangle \]

\[ \langle x_1, \ldots, x_n \mid \varphi_1 \land \cdots \land \varphi_n \rangle \]

Toward game comonads: turn plays into structures

Ehrenfeucht-Fraïssé games

- Play projected on $M$

\[
\langle \rangle \\
\Downarrow \\
\langle x_1 \mid \varphi_1 \rangle \\
\Downarrow \\
\langle x_1, x_2 \mid \varphi_1 \land \varphi_2 \rangle \\
\Downarrow \\
\langle x_1, \ldots, x_n \mid \varphi_1 \land \cdots \land \varphi_n \rangle
\]

$M$ is an element of a structure $\text{REF}(M)$ with carrier $M^+$. 

Toward game comonads: turn plays into structures

Ehrenfeucht-Fraïssé games

► Play projected on $M$

\[
\begin{align*}
\langle \rangle \\
\downarrow \\
\langle x_1 \mid \varphi_1 \rangle \\
\downarrow \\
\langle x_1, x_2 \mid \varphi_1 \land \varphi_2 \rangle \\
\downarrow \\
\langle x_1, \ldots, x_n \mid \varphi_1 \land \cdots \land \varphi_n \rangle
\end{align*}
\]

is an element of a structure $R_{\text{EF}}(M)$ with carrier $M^+$.

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Game comonads: turn plays into structures

Ehrenfeucht-Fraïssé games

- Play projected on $M$ is an element of a structure $R_{\text{EF}}(M)$ with carrier $M^+$. 

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Game comonads: turn plays into structures

Ehrenfeucht-Fraïssé games

- Play projected on $M$ is an element of a structure $R_{EF}(M)$ with carrier $M^+$.

Other examples

- Pebble games.
- Modal fragment, Hybrid fragment, Guarded fragments, . . .

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Game comonads: turn plays into structures

Ehrenfeucht-Fraïssé games

- Play projected on $M$ is an element of a structure $R_{\text{EF}}(M)$ with carrier $M^+$. 

Other examples

- Pebble games.
- Modal fragment, Hybrid fragment, Guarded fragments, …

Adjunctions

- The $R(M)$ are structures with a forest order.

\[
\begin{array}{c}
\text{A} \\
\downarrow \\
\text{Struct}(\sigma)
\end{array}
\]

\[
R
\]

---


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Game comonads: turn plays into structures

Ehrenfeucht-Fraïssé games
- Play projected on $M$ is an element of a structure $R_{\text{EF}}(M)$ with carrier $M^+$.

Other examples
- Pebble games.
- Modal fragment, Hybrid fragment, Guarded fragments, . . .

Adjunctions
- The $R(M)$ are structures with a forest order.
- In each case, $R$ is a right adjoint.
- Comonads on $\text{Struct}(\sigma)$.

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Arboreal categories

Abramsky & Reggio (2021, 2023).

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Arboreal categories: motivations

\[ \mathcal{A} \xrightarrow{L} \text{Struct}(\sigma) \xleftarrow{R} \]

Abramsky & Reggio (2021, 2023).
Arboreal categories: motivations

\[ \mathcal{A} \quad \Downarrow \quad \text{Struct}(\sigma) \quad \Downarrow \quad \mathcal{A} \]

Conditions on \( \mathcal{A} \) which yield well-behaved games.

Abramsky & Reggio (2021, 2023).
Arboreal categories and model comparison games

Arboreal categories: main ideas

Arboreal category $\mathcal{A}$.
Arboreal categories: main ideas

Arboreal category \( A \).

- Factorization system \((\mathcal{Q}, \mathcal{M})\) on \( A \):
  each morphism \( f \) factors as \((e \in \mathcal{Q}, m \in \mathcal{M})\)
  \[
  \begin{array}{ccc}
  \bullet & \xrightarrow{f} & \bullet \\
  \downarrow^{e} & & \downarrow^{m} \\
  \bullet & & \bullet
  \end{array}
  \]

- Typically, the "embeddings" \( m \in \mathcal{M} \) are embeddings of structures which are forest morphisms.

- \( P \in A \) is a path when its \( \mathcal{M} \)-subobjects form a finite chain \( S_1 S_2 \cdots S_n P \).

Induced functor \( A \to \text{Tree} \).

Abramsky & Reggio (2021, 2023).
Arboreal categories and model comparison games

Arboreal categories: main ideas

Arboreal category \( \mathcal{A} \).

- Factorization system \((\mathcal{Q}, \mathcal{M})\) on \( \mathcal{A} \):
  each morphism \( f \) factors as \( (e \in \mathcal{Q}, m \in \mathcal{M}) \)

\[
\bullet \xrightarrow{\; e \;} \bullet \xrightarrow{\; f \;} \bullet \xrightarrow{\; m \;} \bullet
\]

- Typically, the “embeddings” \( m \in \mathcal{M} \) are embeddings of structures which are forest morphisms.
Arboreal categories: main ideas

Arboreal category $\mathcal{A}$.

▶ Factorization system $(\mathcal{Q}, \mathcal{M})$ on $\mathcal{A}$: each morphism $f$ factors as

$$f = e \downarrow \cdot \dashv m \quad (e \in \mathcal{Q}, \ m \in \mathcal{M})$$

▶ Typically, the “embeddings” $m \in \mathcal{M}$ are embeddings of structures which are forest morphisms.

▶ $P \in \mathcal{A}$ is a path when its $\mathcal{M}$-subobjects form a finite chain

$$S_1 \rightarrow S_2 \rightarrow \cdots \rightarrow S_n \leftarrow P$$

Abramsky & Reggio (2021, 2023).

Reggio & Riba (LIP, ENS de Lyon)  Finitely accessible arboreal adjunctions and Hintikka formulae
Arboreal categories: main ideas

**Arboreal category $\mathcal{A}$.**

- Factorization system $(\mathcal{Q}, \mathcal{M})$ on $\mathcal{A}$:
  
  Each morphism $f$ factors as $f = (e \in \mathcal{Q}, m \in \mathcal{M})$.

- Typically, the “embeddings” $m \in \mathcal{M}$ are embeddings of structures which are forest morphisms.

- $P \in \mathcal{A}$ is a path when its $\mathcal{M}$-subobjects form a finite chain $S_1 \cong S_2 \cong \cdots \cong S_n \cong P$.

- Induced functor $\mathcal{A} \rightarrow \text{Tree}$.

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Abramsky & Reggio (2021, 2023).

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Arboreal categories: back-and-forth equivalence

Back-and-forth game $\mathcal{G}(X, Y)$.

$\mathcal{G}(X, Y)$ is a back-and-forth game played by Spoiler and Duplicator.

Definition: $X, Y \in \mathcal{A}$ are back-and-forth equivalent if Duplicator wins $\mathcal{G}(X, Y)$.

Bisimulation via open maps. (Joyal, Nielsen, Winskel)
Arboreal categories: back-and-forth equivalence

**Back-and-forth game** $G(X, Y)$.  
- Positions are spans of “embeddings”

\[ X \xrightarrow{P} Y \]

\[ (X, Y \in \mathcal{A}) \]

\[ (P \text{ path}) \]

Abramsky & Reggio (2021, 2023).

Reggio & Riba (LIP, ENS de Lyon)  
Finitely accessible arboreal adjunctions and Hintikka formulae
Arboreal categories and model comparison games

Arboreal categories: back-and-forth equivalence

**Back-and-forth game** $G(X, Y)$.  

- Positions are spans of “embeddings”  
- Moves: (played by Spoiler and Duplicator)

$X \xleftarrow{P} \xrightarrow{P} Y$  

$(X, Y \in \mathcal{A})$  

$(P \text{ path})$

Definition: $X, Y \in \mathcal{A}$ are back-and-forth equivalent if Duplicator wins $G(X, Y)$.

Bisimulation via open maps. (Joyal, Nielsen, Winskel)

Abramsky & Reggio (2021, 2023).

Reggio & Riba (LIP, ENS de Lyon)  

Finitely accessible arboreal adjunctions and Hintikka formulae
Arboreal categories: back-and-forth equivalence

**Back-and-forth game** $G(X, Y)$. ($X, Y \in A$)

- Positions are spans of “embeddings” ($P$ path)
- Moves: (played by Spoiler and Duplicator)

\[
\begin{tikzpicture}
  \node (X) at (0, 0) {$X$};
  \node (Q) at (1, 0) {$Q$};
  \node (Y) at (2, 0) {$Y$};
  \node (P) at (1, -1) {$P$};

  \draw[->] (X) -- (Q) node[midway, below] (TextNode) {P};
  \draw[->] (Q) -- (Y) node[midway, below] (TextNode) {Q};
  \draw[->] (X) -- (P) node[midway, above] (TextNode) {X};
  \draw[->] (Y) -- (P) node[midway, above] (TextNode) {Y};

  \draw[->] (X) edge[loop below] node {P} (X);
  \draw[->] (Y) edge[loop below] node {Q} (Y);
  \draw[->] (P) edge[loop below] node {X} (P);
\end{tikzpicture}
\]

Definition $X, Y \in A$ are back-and-forth equivalent if Duplicator wins $G(X, Y)$.

- Bisimulation via open maps. (Joyal, Nielsen, Winskel)

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Abramsky & Reggio (2021, 2023).

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Arboreal categories: back-and-forth equivalence

**Back-and-forth game** $G(X, Y)$. ($X, Y \in \mathcal{A}$)

- Positions are spans of “embeddings” ($P$ path)
- Moves: (played by Spoiler and Duplicator)

![Diagram]

Position $P$ moves from $X$ to $Q$ and $Y$.

Definition $X, Y \in \mathcal{A}$ are back-and-forth equivalent if Duplicator wins $G(X, Y)$.

Bisimulation via open maps. (Joyal, Nielsen, Winskel)

Abramsky & Reggio (2021, 2023).
Arboreal categories: back-and-forth equivalence

**Back-and-forth game** $G(X, Y)$.  

- Positions are spans of “embeddings” 
- Moves: (played by Spoiler and Duplicator)

\[
\begin{array}{c}
X \\
\mathop{\downarrow} \mathop{\leftarrow} \mathop{\downarrow} \\
Q & P & Y \\
\mathop{\leftarrow} \mathop{\uparrow} \mathop{\leftarrow} \\
(Spoiler) & & (Duplicator)
\end{array}
\]

or symmetrically.

Definition $X, Y \in \mathcal{A}$ are back-and-forth equivalent if Duplicator wins $G(X, Y)$.

Bisimulation via open maps. (Joyal, Nielsen, Winskel)

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Abramsky & Reggio (2021, 2023).
Arboreal categories: back-and-forth equivalence

Back-and-forth game $G(X, Y)$. (X, Y ∈ A)

- Positions are spans of “embeddings” ($P$ path)
- Moves: (played by Spoiler and Duplicator)

$\xymatrix{ X \ar@<1ex>[r]^-{P} \ar@<1ex>[l]_-{(Spoiler)} & Q \ar@<1ex>[l]^-{(Duplicator)} \\ Y \ar@<1ex>[u] \ar@<1ex>[r] & }$

- Duplicator wins if they can always respond.
Arboreal categories: back-and-forth equivalence

Back-and-forth game $G(X, Y)$. ($X, Y \in A$)

- Positions are spans of “embeddings” ($P$ path)
- Moves: (played by Spoiler and Duplicator)

$\begin{array}{cc}
X & P \\
\text{(Spoiler)} & \downarrow \\
Q & \text{(Duplicator)} \\
\end{array}$

or symmetrically.

- Duplicator wins if they can always respond.

Definition

$X, Y \in A$ are back-and-forth equivalent if Duplicator wins $G(X, Y)$.
Arboreal categories and model comparison games

Arboreal categories: back-and-forth equivalence

**Back-and-forth game** $G(X, Y)$.  

- Positions are spans of “embeddings”  
- Moves: 

  

&middot; Duplicator wins if they can always respond.

**Definition**

$X, Y \in A$ are **back-and-forth equivalent** if Duplicator wins $G(X, Y)$.

- Bisimulation via open maps.  

(Abramsky & Reggio (2021, 2023).

Reggio & Riba (LIP, ENS de Lyon)  
Finitely accessible arboreal adjunctions and Hintikka formulae  

Our goal

\[
\begin{tikzcd}
\mathcal{A} & \mathcal{E} \\
& \downarrow
\end{tikzcd}
\]

\[ L \quad \perp \quad R \]

Example (Ehrenfeucht-Fraïssé games)

Arboreal A with right adjoint R

\[
\text{Struct}(\sigma) \rightarrow A
\]

such that

\[ M, N \text{ are } L_{\infty}(\sigma)\text{-equivalent} \iff R_{EF}(M), R_{EF}(N) \text{ are back-and-forth equivalent} \]

Goal

Give sufficient conditions on L: A \rightleftarrows E: R so that

\[ M, N \in E \text{ are } L_{\infty}\text{-equivalent} \Rightarrow R(M), R(N) \in A \text{ are back-and-forth equivalent} \]
Our goal

\[ \mathcal{A} \sqcup \downarrow \sqcap \sqcup \mathcal{E} \]

Example (Ehrenfeucht-Fraïssé games)

Arboreal \( \mathcal{A} \) with right adjoint \( R_{EF} : \text{Struct}(\sigma) \to \mathcal{A} \) such that

\( M, N \) are \( \mathcal{L}_\infty(\sigma) \)-equivalent \iff

\( R_{EF}(M), R_{EF}(N) \) are back-and-forth equivalent
Our goal

\[ \mathcal{A} \quad \bot \quad \mathcal{E} \]

Example (Ehrenfeucht-Fraïssé games)

Arboreal \( \mathcal{A} \) with right adjoint \( \text{R}_{\text{EF}} : \text{Struct}(\sigma) \to \mathcal{A} \) such that

\[ M, N \text{ are } \mathcal{L}_\infty(\sigma) \text{-equivalent} \iff \text{R}_{\text{EF}}(M), \text{R}_{\text{EF}}(N) \text{ are back-and-forth equivalent} \]

Goal

Give sufficient conditions on \( L : \mathcal{A} \leftrightarrow \mathcal{E} : \text{R} \) so that

\[ M, N \in \mathcal{E} \text{ are } \mathcal{L}_\infty \text{-equivalent} \implies \text{R}(M), \text{R}(N) \in \mathcal{A} \text{ are back-and-forth equivalent} \]
A “structure theorem” for arboreal adjunctions

\[
\begin{array}{c}
\mathcal{A} \\
\bot \\
\mathcal{E}
\end{array}
\]

\[
\begin{array}{c}
L \\
R
\end{array}
\]
A “structure theorem” for arboreal adjunctions

In many examples:
- \( \mathcal{A} \) and \( \mathcal{E} \) are locally finitely presentable,
- the right \( R: \mathcal{E} \to \mathcal{A} \) adjoint is finitary,
A “structure theorem” for arboreal adjunctions

In many examples:

- $\mathcal{A}$ and $\mathcal{E}$ are locally finitely presentable,
- the right $R : \mathcal{E} \to \mathcal{A}$ adjoint is finitary,
- the paths $P$ of $\mathcal{A}$ are finitely presentable,
A “structure theorem” for arboreal adjunctions

In many examples:
- \( \mathcal{A} \) and \( \mathcal{E} \) are locally finitely presentable,
- the right \( R: \mathcal{E} \to \mathcal{A} \) adjoint is finitary,
- the paths \( P \) of \( \mathcal{A} \) are finitely presentable,
- given \( f: P \to X \) in \( \mathcal{A} \),

\[ f \text{ "embedding" in } \mathcal{A} \iff L(f) \text{ embedding of structures in } \mathcal{E} \]
A “structure theorem” for arboreal adjunctions

\[
\begin{array}{c}
\mathcal{A} \\
\perp \\
\mathcal{E}
\end{array}
\xymatrix{
\mathcal{A} \ar@/^/[rr]^{L} \\
\perp \\
\mathcal{E} \ar@/_/[rr]_{R}
}
\]

In many examples:

- \( \mathcal{A} \) and \( \mathcal{E} \) are locally finitely presentable,
- the right \( R: \mathcal{E} \to \mathcal{A} \) adjoint is finitary,
- the paths \( P \) of \( \mathcal{A} \) are finitely presentable,
- given \( f: P \to X \) in \( \mathcal{A} \),

\[
f \text{“embedding” in } \mathcal{A} \iff L(f) \text{ embedding of structures in } \mathcal{E}
\]

**Theorem (Reggio & Riba)**

\[ M, N \in \mathcal{E} \text{ are } L_\infty(\mathcal{E})\text{-equivalent} \implies R(M), R(N) \in \mathcal{A} \text{ are back-and-forth equivalent} \]
Proof

\[ \mathcal{A} \xrightarrow{\perp} \xleftarrow{R} \mathcal{E} \]

- \( \mathcal{E} \) and \( \mathcal{A} \) locally finitely presentable,
- finitary right-adjoint \( R: \mathcal{E} \to \mathcal{A} \),
- paths \( P \) of \( \mathcal{A} \) finitely presentable.

\( f: P \to X \) “embedding” in \( \mathcal{A} \) \iff \( L(f) \) embedding of structures in \( \mathcal{E} \).
Proof

\[ \mathcal{A} \rightleftharpoons_{\perp}^{L} \mathcal{E} \rightleftharpoons_{R}^{\mathcal{A}} \]

- $\mathcal{E}$ and $\mathcal{A}$ locally finitely presentable,
- finitary right-adjoint $R: \mathcal{E} \to \mathcal{A}$,
- paths $P$ of $\mathcal{A}$ finitely presentable.

- $f: P \to X$ “embedding” in $\mathcal{A} \iff L(f)$ embedding of structures in $\mathcal{E}$.
- $\mathcal{A}$ and $\mathcal{E}$ categories of models of (cartesian) theories. (Coste 1976)
Proof

- $\mathcal{A}$ and $\mathcal{E}$ locally finitely presentable,
- finitary right-adjoint $R: \mathcal{E} \to \mathcal{A}$,
- paths $P$ of $\mathcal{A}$ finitely presentable.

- $f: P \to X$ “embedding” in $\mathcal{A}$ $\iff$ $L(f)$ embedding of structures in $\mathcal{E}$.
- $\mathcal{A}$ and $\mathcal{E}$ categories of models of (cartesian) theories. (Coste 1976)
- Embeddings of structures in $\mathcal{E}$ (of f.p. domain) are definable in $\mathcal{L}_\infty(\mathcal{E})$.
  (functorial semantics and Yoneda lemma)
Proof

- $\mathcal{E}$ and $\mathcal{A}$ locally finitely presentable,
- finitary right-adjoint $R: \mathcal{E} \to \mathcal{A}$,
- paths $P$ of $\mathcal{A}$ finitely presentable.

- $f: P \to X$ “embedding” in $\mathcal{A} \iff L(f)$ embedding of structures in $\mathcal{E}$.
- $\mathcal{A}$ and $\mathcal{E}$ categories of models of (cartesian) theories.
- Embeddings of structures in $\mathcal{E}$ (of f.p. domain) are definable in $L_\infty(\mathcal{E})$.
  (functorial semantics and Yoneda lemma)
- Left adjoint $L: \mathcal{A} \to \mathcal{E}$ induces a formula translation $L_\infty(\mathcal{E}) \to L_\infty(\mathcal{A})$.
  (Hodges’ word-constructions (1974, 1975))
**Proof**

- \( \mathcal{E} \) and \( \mathcal{A} \) locally finitely presentable,
- finitary right-adjoint \( R: \mathcal{E} \to \mathcal{A} \),
- paths \( P \) of \( \mathcal{A} \) finitely presentable.

\[ \begin{array}{c}
\mathcal{A} \\
\downarrow L \\
\mathcal{E} \\
\mathcal{E} \\
\uparrow R \\
\mathcal{A}
\end{array} \]

- \( f: P \to X \) “embedding” in \( \mathcal{A} \) \( \iff \) \( L(f) \) embedding of structures in \( \mathcal{E} \).
- \( \mathcal{A} \) and \( \mathcal{E} \) categories of models of (cartesian) theories. \( \text{(Coste 1976)} \)
- Embeddings of structures in \( \mathcal{E} \) (of f.p. domain) are definable in \( \mathcal{L}_\infty(\mathcal{E}) \).
  \( \text{(functorial semantics and Yoneda lemma)} \)
- Left adjoint \( L: \mathcal{A} \to \mathcal{E} \) induces a formula translation \( \mathcal{L}_\infty(\mathcal{E}) \to \mathcal{L}_\infty(\mathcal{A}) \).
  \( \text{(Hodges’ word-constructions (1974, 1975))} \)
- Hintikka formulae in \( \mathcal{L}_\infty(\mathcal{A}) \) for back-and-forth games in \( \mathcal{A} \).
  \( \text{(define ordinal ranks of positions in games)} \)
Proof

\[ \mathcal{A} \leftrightarrow \Downarrow L \leftrightarrow R \rightarrow \mathcal{E} \]

- \( \mathcal{E} \) and \( \mathcal{A} \) locally finitely presentable,
- finitary right-adjoint \( R : \mathcal{E} \rightarrow \mathcal{A} \),
- paths \( P \) of \( \mathcal{A} \) finitely presentable.

- \( f : P \rightarrow X \) “embedding” in \( \mathcal{A} \) \( \iff \) \( L(f) \) embedding of structures in \( \mathcal{E} \).
- \( \mathcal{A} \) and \( \mathcal{E} \) categories of models of (cartesian) theories. (Coste 1976)
- Embeddings of structures in \( \mathcal{E} \) (of f.p. domain) are definable in \( \mathcal{L}_\infty(\mathcal{E}) \).
  (functorial semantics and Yoneda lemma)
- Left adjoint \( L : \mathcal{A} \rightarrow \mathcal{E} \) induces a formula translation \( \mathcal{L}_\infty(\mathcal{E}) \rightarrow \mathcal{L}_\infty(\mathcal{A}) \).
  (Hodges’ word-constructions (1974, 1975))
- Hintikka formulae in \( \mathcal{L}_\infty(\mathcal{A}) \) for back-and-forth games in \( \mathcal{A} \).
  (define ordinal ranks of positions in games)

Lemma

If \( X, Y \) are equivalent in \( \mathcal{L}_\infty(\mathcal{A}) \), then \( X, Y \) are back-and-forth equivalent in \( \mathcal{A} \).
Proof

$\mathcal{A} \quad \perp \quad \mathcal{E}$

- $\mathcal{E}$ and $\mathcal{A}$ locally finitely presentable,
- finitary right-adjoint $R: \mathcal{E} \to \mathcal{A}$,
- paths $P$ of $\mathcal{A}$ finitely presentable.

- $f: P \to X$ “embedding” in $\mathcal{A} \iff L(f)$ embedding of structures in $\mathcal{E}$.
- $\mathcal{A}$ and $\mathcal{E}$ categories of models of (cartesian) theories. (Coste 1976)
- Embeddings of structures in $\mathcal{E}$ (of f.p. domain) are definable in $L_\infty(\mathcal{E})$.
  (functorial semantics and Yoneda lemma)
- Left adjoint $L: \mathcal{A} \to \mathcal{E}$ induces a formula translation $L_\infty(\mathcal{E}) \to L_\infty(\mathcal{A})$.
  (Hodges’ word-constructions (1974, 1975))
- Hintikka formulae in $L_\infty(\mathcal{A})$ for back-and-forth games in $\mathcal{A}$.
  (define ordinal ranks of positions in games)

Lemma

If $X, Y$ are equivalent in $L_\infty(\mathcal{A})$, then $X, Y$ are back-and-forth equivalent in $\mathcal{A}$.

- $R: \mathcal{E} \to \mathcal{A}$ induces a formula translation $L_\infty(\mathcal{A}) \to L_\infty(\mathcal{E})$. 
Proof

- \( \mathcal{E} \) and \( \mathcal{A} \) locally finitely presentable,
- finitary right-adjoint \( R : \mathcal{E} \to \mathcal{A} \),
- paths \( P \) of \( \mathcal{A} \) finitely presentable.
- \( f : P \to X \) “embedding” in \( \mathcal{A} \) \( \iff \) \( L(f) \) embedding of structures in \( \mathcal{E} \).
- \( \mathcal{A} \) and \( \mathcal{E} \) categories of models of (cartesian) theories. (Coste 1976)
- Embeddings of structures in \( \mathcal{E} \) (of f.p. domain) are definable in \( \mathcal{L}_\infty(\mathcal{E}) \).
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- Left adjoint \( L : \mathcal{A} \to \mathcal{E} \) induces a formula translation \( \mathcal{L}_\infty(\mathcal{E}) \to \mathcal{L}_\infty(\mathcal{A}) \).
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Theorem

If \( M, N \) are equivalent in \( \mathcal{L}_\infty(\mathcal{E}) \), then \( R(M), R(N) \) are back-and-forth equivalent in \( \mathcal{A} \).
An application

\[ \mathcal{A} \xrightarrow{L} \text{Struct}(\sigma) \xleftarrow{R} \]

**Theorem**

\[ M, N \in \text{Struct}(\sigma) \text{ are } \mathcal{L}_\infty(\sigma)-equivalent \quad \implies \quad R(M), R(N) \in \mathcal{A} \text{ are back-and-forth equivalent} \]
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**Example.**

- \((\mathbb{Q}, <)\) and \((\mathbb{R}, <)\) are \(\mathcal{L}_\infty(<)\)-equivalent.
- \(R(\mathbb{Q}), R(\mathbb{R})\) are back-and-forth equivalent in \(\mathcal{A}\).

**Remark.**

- Many non-isomorphic \(\mathcal{L}_\infty(<)\)-equivalent structures.
An application

$\mathcal{A} \quad \perp \quad \text{Struct}(\sigma)$

$M, N \in \text{Struct}(\sigma)$ are $\mathcal{L}_\infty(\sigma)$-equivalent $\implies$ $R(M), R(N) \in \mathcal{A}$ are back-and-forth equivalent

Example.

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Remark.

- Many non-isomorphic $\mathcal{L}_\infty(\sigma)$-equivalent structures.

Game comonad for MSO. (Jackl, Marsden & Shah, 2022)

- $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$ are not MSO$(\sigma)$-equivalent.
Conclusion and future work

Toward a structure theory of game comonads via arboreal categories.

- General conditions on $R : \mathcal{E} \to \mathcal{A}$ for
  
  $M, N \in \mathcal{E}$ are $\mathcal{L}_\infty(\mathcal{E})$-equivalent $\implies$ $R(M), R(N) \in \mathcal{A}$ are back-and-forth equivalent

- Restricts to finite games and finitary logic.
- Covers various examples.

Future work.
- Higher presentability ranks.
  (Lindström quantifiers (via the games of (Caicedo 1980)))
  (Comonadic modal logic)
- Convey stronger invariants?
  (E.g. finite variable constraint for pebble games)

Thanks for your attention!
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