Finitely accessible arboreal adjunctions and Hintikka formulae

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A reformulation of an old example (1/2)

Formulation inspired from Abramsky & Shah (2018, 2021).

A reformulation of an old example (1/2)

Situation

$$\langle \overline{x} \mid \varphi \rangle$$

where

- $\overline{x} = x_1, \ldots, x_n$
- φ finite conjunction of contraints $((x_i < x_j)$ or $(x_i = x_j))$

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A reformulation of an old example (1/2)

Situation

$$\langle \overline{x} \mid \varphi \rangle \overset{m}{\rightarrowtail} (M, <_M)$$

where

- $\overline{x} = x_1, \ldots, x_n$
- φ finite conjunction of contraints $((x_i < x_j) \text{ or } (x_i = x_j))$
- m is an order embedding:

$$\begin{aligned} m(x_i) <_M m(x_j) \quad &\iff \quad (x_i < x_j) \text{ in } \varphi \\ m(x_i) = m(x_j) \quad &\iff \quad (x_i = x_j) \text{ in } \varphi \end{aligned}$$

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A reformulation of an old example (2/2)

Let $(M, <_M)$ and $(N, <_N)$ be dense linear orders without end points.

(e.g. $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$)

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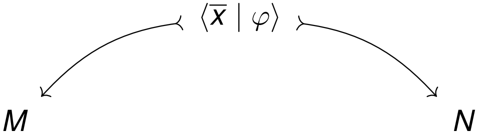

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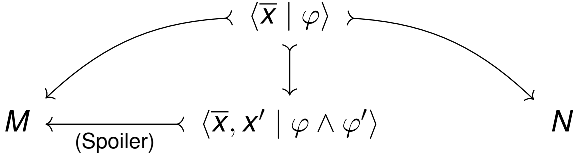

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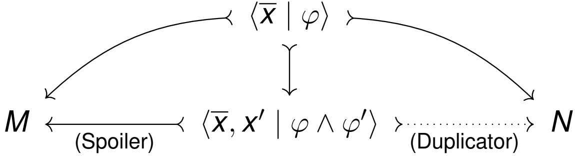

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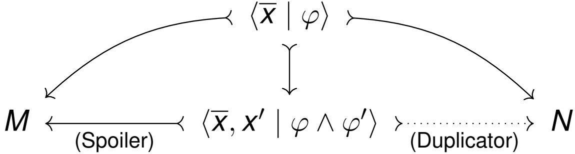

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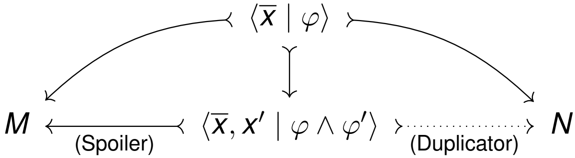

or symmetrically.

- Duplicator wins since they can always respond.

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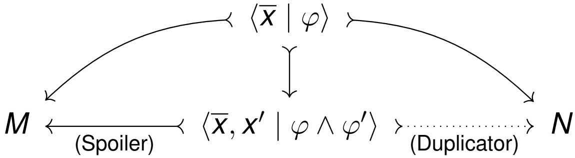

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Corollary (. . . , Karp (1965))

$(M, <_M)$ and $(N, <_N)$ are equivalent in $\mathcal{L}_\infty(<)$.

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Toward game comonads

Abramsky, Dawar & Wang (2017), Abramsky & Shah (2018, 2021)...

Toward game comonads: turn plays into structures

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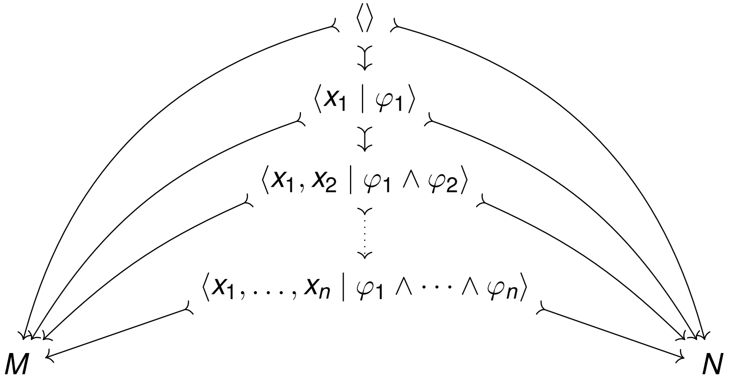
Ehrenfeucht-Fraïssé games

- Play



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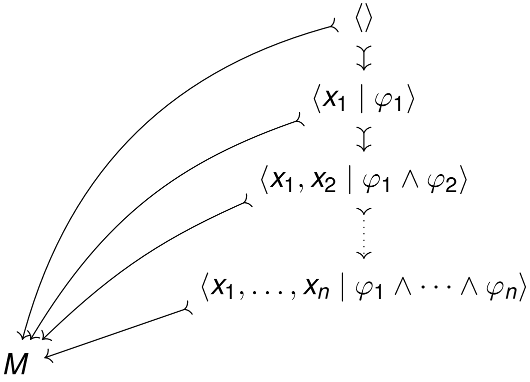

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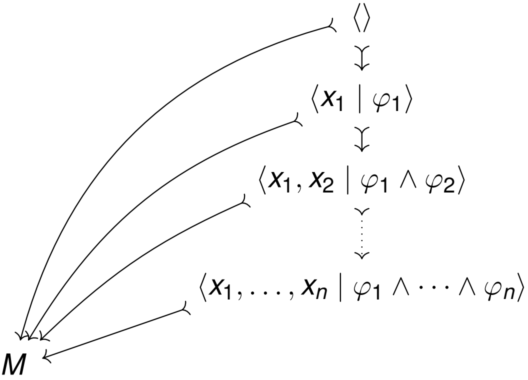

is an element of a structure $R_{\mathbb{EF}}(M)$ with carrier M^+.

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Other examples

- Pebble games.
- Modal fragment, Hybrid fragment, Guarded fragments, …

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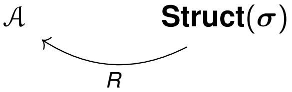
Adjunctions

- The $R(M)$ are structures with a forest order.



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Adjunctions

- The $R(M)$ are structures with a forest order.
- In each case, R is a right adjoint.
- Comonads on $\mathbf{Struct}(\sigma)$.

$$\mathcal{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathbf{Struct}(\sigma) \qquad (L \dashv R, \text{ shown as } \bot)$$

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Arboreal categories

Abramsky & Reggio (2021, 2023).

Arboreal categories: motivations

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Arboreal categories: motivations

$$\mathcal{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathbf{Struct}(\sigma) \quad (L \dashv R)$$

Conditions on $\mathcal{A}$ which yield well-behaved games.

Abramsky & Reggio (2021, 2023).

Arboreal categories: main ideas

Arboreal category $\mathcal{A}$.

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- Factorization system $(\mathcal{Q}, \mathcal{M})$ on $\mathcal{A}$:
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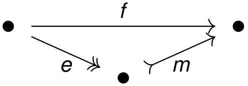

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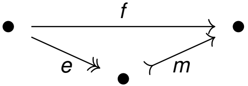

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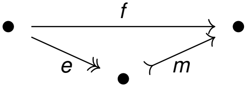

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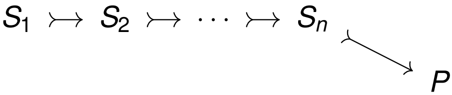

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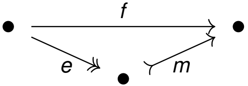

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$$S_1 \rightarrowtail S_2 \rightarrowtail \cdots \rightarrowtail S_n \rightarrowtail P$$

- Induced functor $\mathcal{A} \to$ **Tree**.

Abramsky & Reggio (2021, 2023).

Arboreal categories: back-and-forth equivalence

Back-and-forth game $\mathcal{G}(X, Y)$. $(X, Y \in \mathcal{A})$

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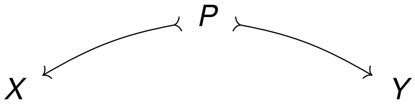

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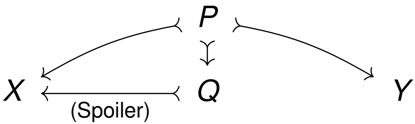

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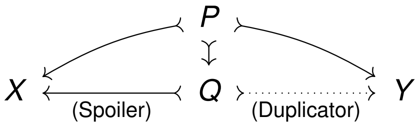

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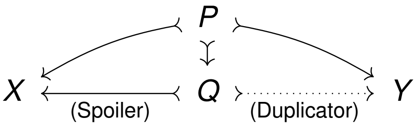

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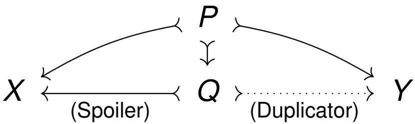

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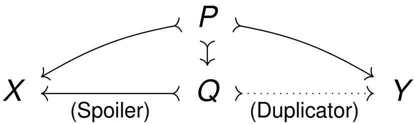

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$X, Y \in \mathcal{A}$ are back-and-forth equivalent if Duplicator wins $\mathcal{G}(X, Y)$.

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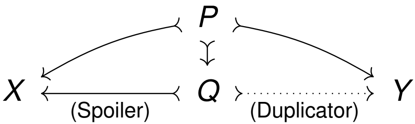

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- Bisimulation via open maps. (Joyal, Nielsen, Winskel)

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Our goal

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Example (Ehrenfeucht-Fraïssé games)

Arboreal $\mathcal{A}$ with right adjoint $R_{\mathbb{EF}} \colon \mathbf{Struct}(\boldsymbol{\sigma}) \to \mathcal{A}$ such that

M, N are $\mathcal{L}_\infty(\boldsymbol{\sigma})$-equivalent $\iff$ $R_{\mathbb{EF}}(M), R_{\mathbb{EF}}(N)$ are back-and-forth equivalent

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Goal

Give sufficient conditions on $L \colon \mathcal{A} \rightleftarrows \mathcal{E} : R$ so that

$M, N \in \mathcal{E}$ are $\mathcal{L}_\infty$-equivalent $\implies$

$R(M), R(N) \in \mathcal{A}$ are back-and-forth equivalent

A “structure theorem” for arboreal adjunctions

$$\mathcal{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{E} \qquad (L \dashv R)$$

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$$\mathcal{A} \underset{R}{\overset{L}{\underset{\bot}{\rightleftarrows}}} \mathcal{E}$$

In many examples:

- $\mathcal{A}$ and $\mathcal{E}$ are locally finitely presentable,
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Theorem (Reggio & R)

$M, N \in \mathcal{E}$ are $\mathcal{L}_\infty(\mathcal{E})$-equivalent $\implies$ $R(M), R(N) \in \mathcal{A}$ are back-and-forth equivalent

Proof

$$\mathcal{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{E} \qquad (L \dashv R)$$

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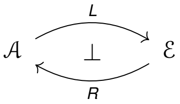

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- Hintikka formulae in $\mathcal{L}_\infty(\mathcal{A})$ for back-and-forth games in $\mathcal{A}$.
  (define ordinal ranks of positions in games)

Proof

$$\mathcal{A} \underset{R}{\overset{L}{\underset{\bot}{\rightleftarrows}}} \mathcal{E}$$

- $\mathcal{E}$ and $\mathcal{A}$ locally finitely presentable,
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Lemma

If X, Y are equivalent in $\mathcal{L}_\infty(\mathcal{A})$, then X, Y are back-and-forth equivalent in $\mathcal{A}$.

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Theorem

If M, N are equivalent in $\mathcal{L}_\infty(\mathcal{E})$, then $R(M), R(N)$ are back-and-forth equivalent in $\mathcal{A}$.

An application

$$\mathcal{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathbf{Struct}(\sigma) \qquad (L \dashv R)$$

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$M, N \in \mathbf{Struct}(\sigma)$ *are* $\mathcal{L}_\infty(\sigma)$*-equivalent* $\Longrightarrow$

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Example.

- $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$ are $\mathcal{L}_\infty(<)$-equivalent.
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- Many non-isomorphic $\mathcal{L}_\infty(<)$-equivalent structures.

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Game comonad for MSO. (Jackl, Marsden & Shah, 2022)

- $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$ are not **MSO**$(<)$-equivalent.

Conclusion and future work

Toward a structure theory of game comonads via arboreal categories.

- General conditions on $R\colon \mathcal{E} \to \mathcal{A}$ for

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Thanks for your attention!